Alireza Mousavi
Brunel University

Embedded Systems and Industrial Controller EE5563 (3)

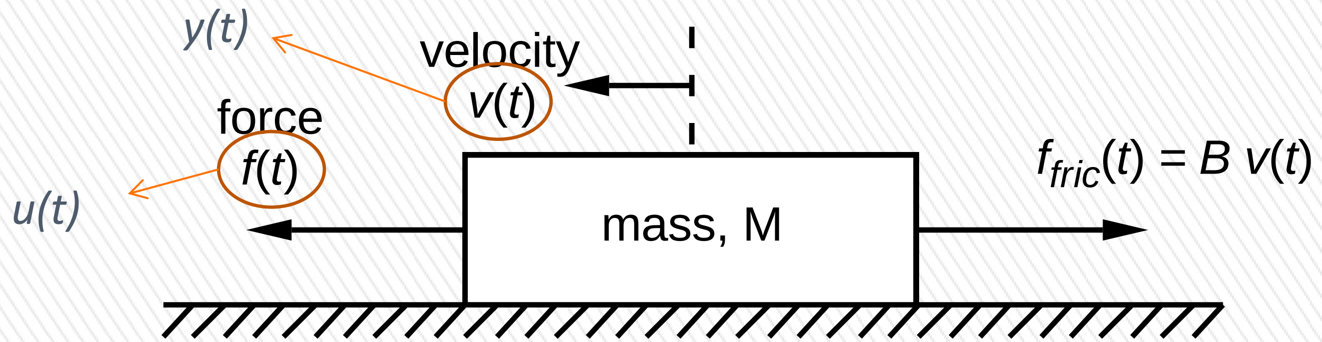
Today

- Transfer Function
- Modelling of Dynamical Systems

Transfer Function

- Transfer function is a dynamical model represented in the Laplace domain.
- In specific the transfer function *G(s)* of a linear, time invariant, single-input single-output system is given by the ration of the Laplace transformed output to the Laplace transformed input;

$$G(s) = \frac{Y(s)}{U(s)} \quad \text{initial condition} = 0 \qquad (1)$$



» The linear dynamical equation: $\frac{dv}{dt} + \frac{B}{M} v(t) = \frac{1}{M} f(t)$

» Let M = 1000 kg and B = 50 Ns/m: $\frac{dv}{dt} + 0.05 v(t) = 0.001 f(t)$

» Taking Laplace Transforms and assuming zero initial conditions:

$$sV(s) + 0.05V(s) = 0.001F(s)$$

So

$$V(s) = \frac{0.001}{s + 0.05} \times F(s)$$

Relates output velocity *Y(s)* to input force *U(s) or the* ***G(s) transfer function***

Example

» Supposing that the Laplace transform of a particular input *u(t)* is infinite,

» Then the corresponding Laplace output *y(t)* will also be infinite,

» But the **transfer function** itself will be finite.

Consider a system described by LDE:

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^n} + \cdots a_0 y = b_0 u + b_1 \frac{du}{dt} + \cdots + b_m \frac{d^m u}{dt^m} \qquad (2)$$

Taking Laplace transforms and assuming that initial condition = 0, the transfer function will be:

$$\left(s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0\right)Y(s)$$

$$= \left(b_m s^m + b_{m-1}s^{m-1} + \ldots + b_1 s + b_0\right)U(s) \qquad (3)$$

The General formulation

Or:

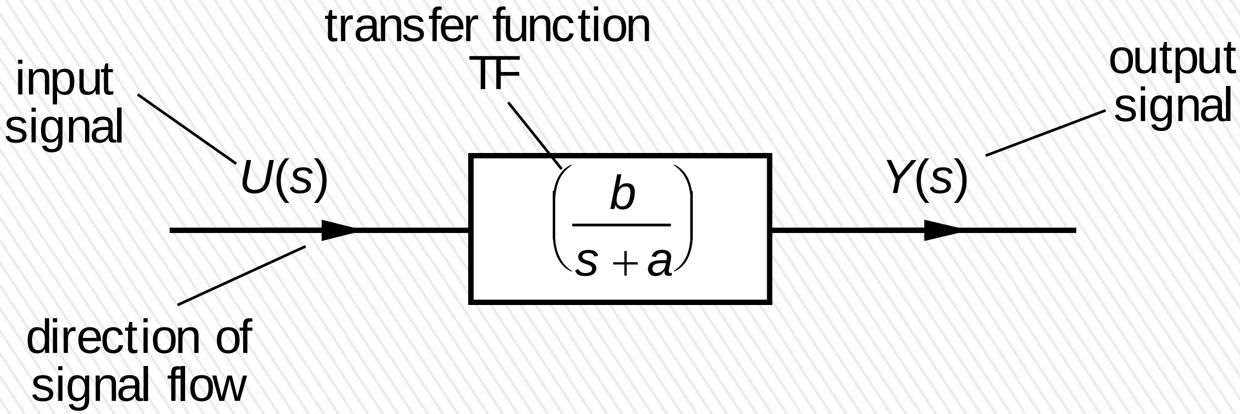
$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_0 + b_1 s + \cdots b_m s^m}{a_0 + a_1 s + \cdots a_n s^n} \qquad (4)$$

» The transfer function is defined **only**
  - with respect to the Laplace transformed equation
  - for zero initial conditions

» It is extremely useful in building a model for a complex system in terms of the individual models for its component parts.
  - a TF (transfer function) is independent of actual input applied
  - it characterizes the **system** itself.

Contd.

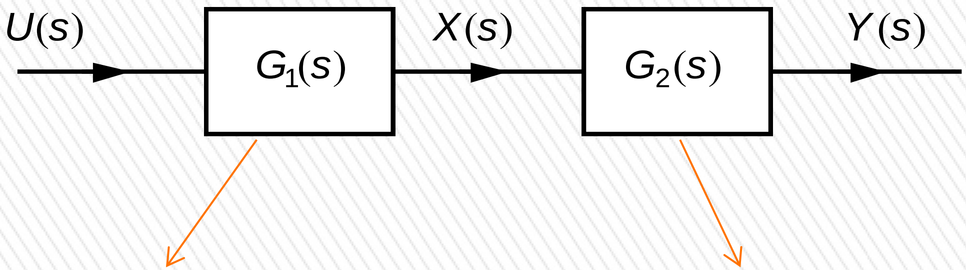
» The building blocks for a First order example



Block Diagram Representation

» Simple block diagram manipulation:

Consider a cascade connection



Transfer function for Equation 1

Transfer function for Equation 2

Then $X(s) = G_1(s)U(s)$ and $Y(s) = G_2(s)X(s)$

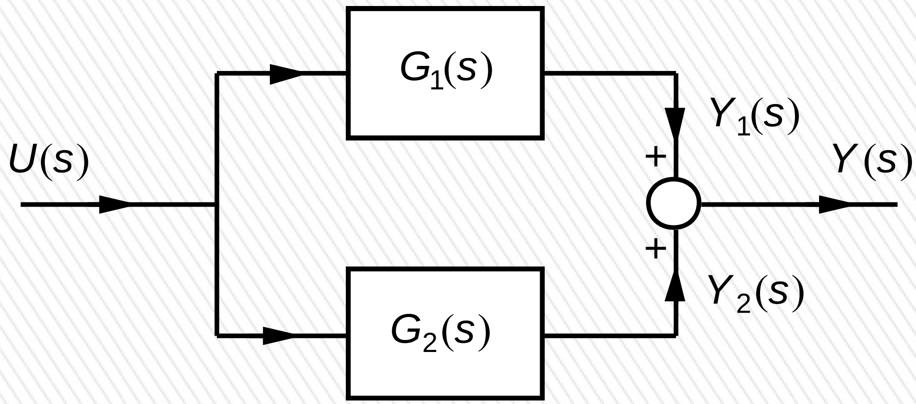
$$\therefore Y(s) = G_1(s)G_2(s)U(s)$$

And then overall Transfer function is:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{G_1(s)G_2(s)U(s)}{U(s)} = G_1(s)G_2(s) \qquad (5)$$

Block Diagram Representation

» Parallel Connection



$$Y(s) = Y_1(s) + Y_2(s)$$

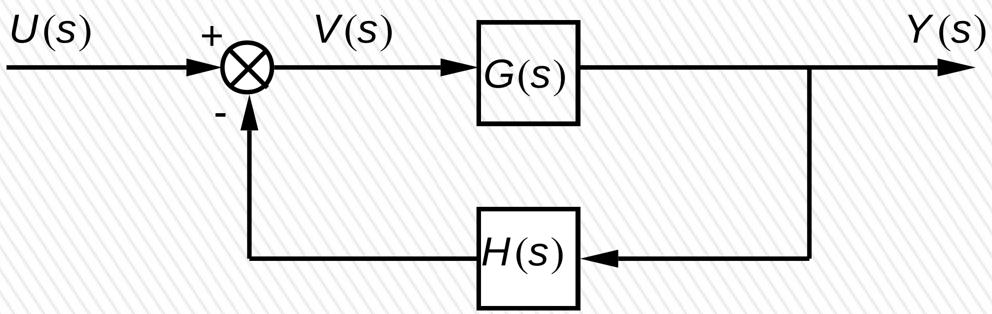
And

$$Y_1(s) = G_1(s)U(s), \qquad Y_2(s) = G_2(s)U(s)$$

Therefore; $$G(s) = \frac{G_1(s)U(s) + G_2(s)U(s)}{U(s)} = G_1(s) + G_2(s) \qquad (6)$$

Block Diagram Representation

» Feedback connection



$$Y(s) = G(s)V(s)$$

$$V(s) = U(s) - H(s)Y(s)$$

$$\therefore \frac{Y(s)}{U(s)} = \frac{G(s)V(s)}{V(s) + H(s)G(s)V(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

Block Diagram Representation

Frequency Domain Models

- Any transfer function is defined as the ratio of Output to Input
- If the output and input are expressed in the **frequency domain** – the frequency transfer function is represented by the **Fourier Transform** function of the ***output to the input***.
- Frequency domain parameters are useful for the analysis, design, control and testing of electro-mechanical systems
- The signal waveform derived from such systems can be interpreted and presented as a series of **sinusoidal** components

» Recall equations 2-4, *time domain and Transfer Function in Laplace domain* (slides 5 & 6)

» **Response to a harmonic Input:**

> *If a harmonic (sinusoidal) input is given as:*

$$u = u_0(cos\omega t + jsin\omega t) \quad (7)$$

Once in steady state, the output will be harmonic as:

$$y = y_0 e^{j\omega t} = y_0(cos\omega t + jsin\omega t) \quad (8)$$

By substituting equations 7 and 8 in equation 2 and cancelling $e^{j\omega t}$ on both sides:

$$y_0 = \left[\frac{b_m(j\omega)^m + b_{m-1}(j\omega)^{m-1} + \cdots + b_0}{a_n(j\omega)^n + a_{n-1}(j\omega)^{n-1} + \cdots + a_0}\right] u_0 \quad (9)$$

Frequency Transfer Function

» And with respect to equation 4:

$$y_0 = G(j\omega)u_0 \quad (10)$$

Bear in mind: $\frac{de^{j\omega t}}{dt} = j\omega e^{j\omega t}$

Here the frequency transfer function (frequency response function) is given by:

$$G(j\omega) = G(s)| = \frac{b_0+b_1(j\omega)+\cdots b_m(j\omega)^m}{a_0+a_1(j\omega)+\cdots+a_n(j\omega)^n} \quad (11)$$

Where $s = j\omega$

The angular frequency $\omega = 2\pi f$ *f is cyclic frequency (Hz)*

the Laplace domain

$$G(j\omega) = \frac{Y(j\omega)}{U(j\omega)} \quad (12)$$

The Fourier transform operators: $Y(j\omega) = F_y(t) \quad and \quad U(j\omega) = F(t)$

Response to harmonic input

» Here we will discuss some dynamical systems that their behaviour can be described with LDE.

Mechanical Systems

Newton Law – Conservation of energy (also briefly discussed in Lecture 1)

1. If net force on the body = 0 and the acceleration = 0

Then the linear momentum is conserved:

$$\sum F = ma$$

2. Action and reaction are equal and opposite : $F_{BA} = -F_{AB}$

Modelling of Dynamical Systems / Mechanical Example

- Variables are: Force and Displacement (Velocity)
- In a spring you have:

$$f_k(t) = Kx(t)$$

Stiffness constant

- Assuming linearity (i.e. Hook's Law) – Laplace representation:

$$F_s = KX(s)$$

- The Damper function: $f_B(t) = Bdv = B\frac{dx}{dt}$

Damper constant

Assuming Linearity, then …

$$F_B(s) = BV(s) = BsX(s) \ \textit{for initial condition IC=0}$$

» Mass, the motion *M* by Newton's Second Law:

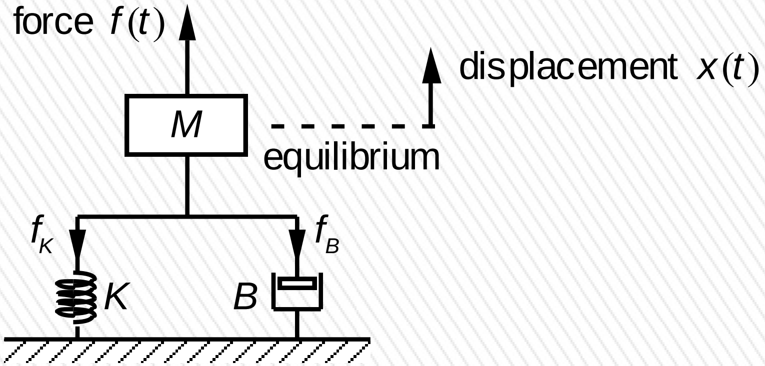
Laplace representation: $F_M(s) = MsV(s) = Ms^2X(s)$ for IC=0

M and K are associated with energy storage $\begin{cases} M \ \ is\ kinetic \\ K\ is\ potential \end{cases}$

And *B with energy dissipation (recall previous lecture)*

Modelling of Dynamical Systems / Mechanical Example

Example Mechanical System



Net force on mass upwards = $F(s) - \left(F_K(s) + F_B(s)\right)$

Apply Newton 2nd Law: $F(s) - K\,X(s) - Bs\,X(s) = Ms^2\,X(s)$

Or
$$\left(s^2 + \frac{B}{M}s + \frac{K}{M}\right)X(s) = \frac{1}{M}F(s)$$

The transfer function:
$$G(s) = \frac{X(s)}{F(s)} = \frac{\frac{1}{M}}{s^2 + \frac{B}{M}s + \frac{K}{M}}$$

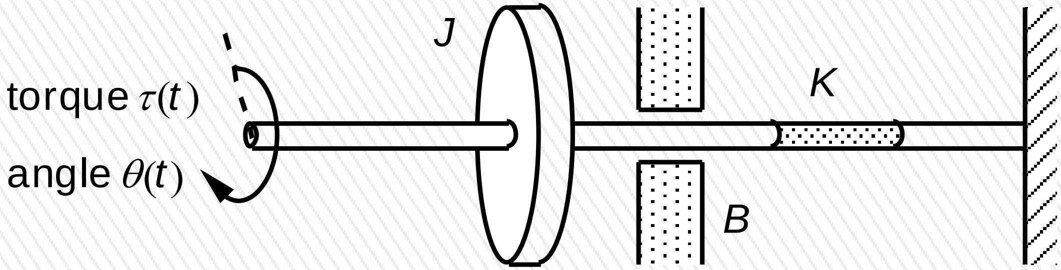
This represents a 2nd order LDE:
$$\frac{d^2x}{dt^2} + \frac{B}{M}\frac{dx}{dt} + \frac{K}{M}x(t) = \frac{1}{M}f(t)$$

When *B=0 (no damping)* $\Rightarrow \frac{d^2x}{dt^2} + \frac{K}{M}x(t) = \frac{1}{M}f(t)$

Simple Harmonic Motion (SHM) when $f(t) = 0, x(0) \neq 0$

Example Mechanical System

» In the case of rotational functions:



J is the moment of inertia (kgm^2)

B is the linear rotational damping $\tau_B(t) = B\dfrac{d\theta}{dt}$

For IC=0 $\Rightarrow$ $\tau_B(s) = Bs\,\theta(s)$

$$\tau_K(t) = K\,\theta(t) \;\Rightarrow\; \tau_K(s) = K\,\theta(s)$$

K is the torsional stiffness

net torque applied = rate of change of angular momentum $= \dfrac{d}{dt}\left(J\dfrac{d\theta}{dt}\right) = J\dfrac{d^2\theta}{dt^2}$

Assuming inertia is constant

Rotational Functions

» in terms of Laplace, net torque $= Js^2\,\theta(s)$ assuming IC=0

net torque acting on $J(s) = \tau(s) - ((\tau_K(s) + \tau_B(s))$

$\therefore \quad Js^2\,\theta(s) = \tau(s) - K\,\theta(s) - Bs\,\theta(s)$ or $\left(s^2 + \frac{B}{J}s + \frac{K}{J}\right)\theta(s) = \frac{1}{J}\tau(s)$

Rotational Functions

Electrical Systems

- Kirchhoff's Laws conservation of energy:
- Key variables are Voltage and Current

KVL : $\sum Voltage\ round\ closed\ loop = 0$

KCL : $\sum Current\ into\ a\ node = 0$

The three factors of Resistor *(R)*, Inductor *(L)*, and Capacitor *(C)*:

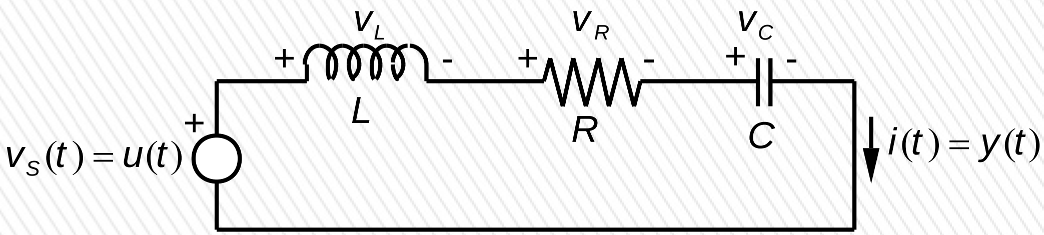
R: $v_R(t) = Ri_R(t) \rightarrow V_R(s) = RI_R(s)$

L: $v_L(t) = L\frac{di_L}{dt} \rightarrow V_L(s) = LsI_L(s) \quad provided\ IC = 0$

C: $v_c(t) = \frac{1}{C}\int i_c dt \rightarrow V_c(s) = \frac{1}{Cs}I_c(s)\ \ provided\ IC = 0$

Observe that multiplying by *s* represents differentiation, dividing by *s* represents integration or in other words $\frac{1}{s}$ is the integrator.

» Example



Apply KVL: $V_s(s) - V_L(s) - V_R(s) - V_c(s) = 0$

$$\therefore V_s(s) - LsI(s) - RI(s) - \frac{1}{Cs}(s) = 0$$

Multiply by *s:* $sV(s) - Ls^2I(s) - sRI(s) - \frac{1}{C}I(s) = 0$

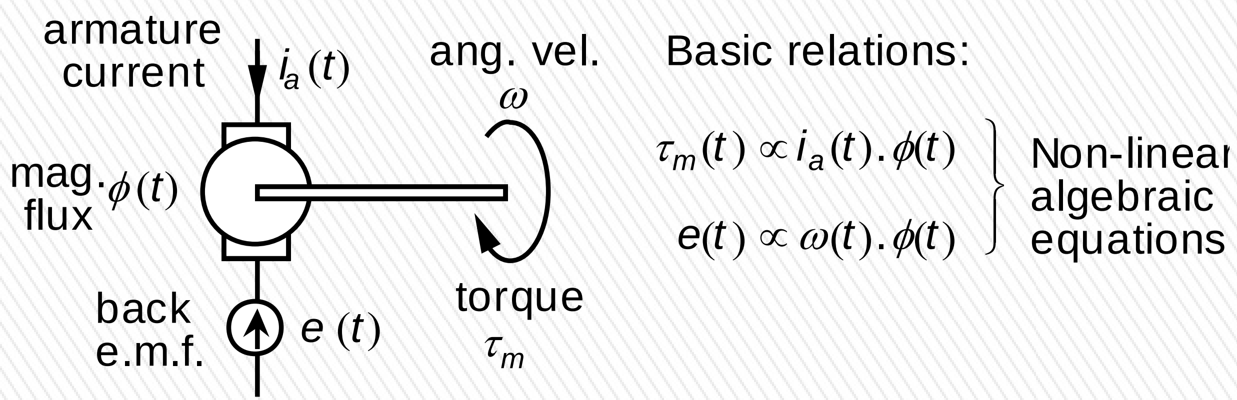
$$\left(Ls^2 + Rs + \frac{1}{C}\right)I(s) = sV_s(s)$$

Therefore the TF, $G(s) = \frac{I(s)}{V_S(s)} = \frac{\frac{1}{L}s}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$ $\quad$ 2nd order system

Series RLC Circuit

» Fundamentals: Conservation of energy (magnetic field and Newton's Law)

A DC Motor:



Therefore: $\tau_m(s) = kI_a(s)\varphi(s) \quad and \quad E(s) = k\omega(s)\varphi(s)$

Now consider a motor with a permanent magnetic field i.e. φ *is constant*

Therefore,

$$\tau_m(s) = cI_a(s) \quad and \quad E(s) = c\omega(s) \quad where \;\; c = k\varphi$$

The constant flux constraint results in a Linear algebraic equations

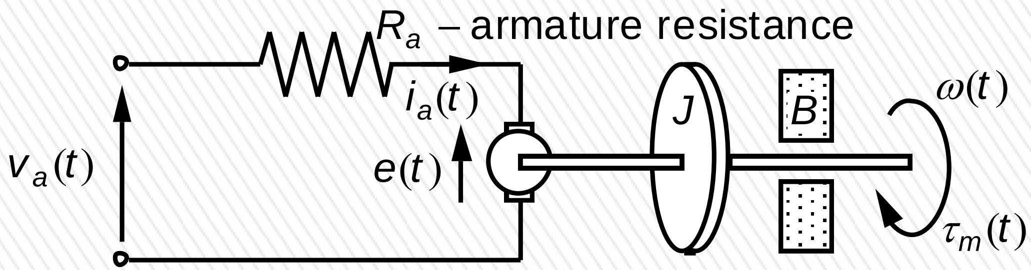
Electromechanical System

Electromechanical Example cont.

» Consider the motor to be driving a mechanical load with *J* and *B* factors

The torque demand of load : $\tau_L(s) = Js^2\theta(s) + Bs\theta(s) = Js\omega(s) + B\omega(s)$

But $\tau_L(s) = \tau_m(s)$ thus $\therefore \left(s + \frac{B}{J}\right)\omega(s) = \frac{c}{J} I_a(s)$ 1st order system



Then $I_a(s) = \dfrac{V_a(s) - E(s)}{R_a} = \dfrac{V_a(s) - c.\omega(s)}{R_a}$

So $\left(s + \dfrac{1}{J}\left(B + \dfrac{c^2}{R_a}\right)\right)\omega(s) = \dfrac{c}{JR_a}V_a(s)$ if armature inductance L_a is included then 2nd order

- Fluid Flow Systems
  - Fundamentals: Conservation of energy (common sense and Bernoulli theorem)
- Thermal Systems
  - Fundamental: Conservation of energy (heat balance)

Further reading