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Embedded Systems and Industrial Controller EE5563 (4)

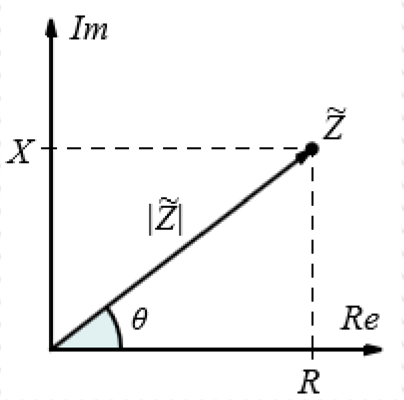
Topics

- More on Transfer Functions of Electromechanical systems
- Properties of Transfer Functions
- Transmissibility Function
- Response Analysis (A prelude to Analytical and Numerical Solutions)

Recall Last week we discussed electrical (RLC) and mechanical (Force and Disposition), and electromechanical systems (force-current) fundamentals.

- Impedance is a Transfer Function relevant to both mechanical and electrical systems (inverse of mobility).

- Impedance extends the concept of resistance to AC circuits, and has both magnitude and phase.

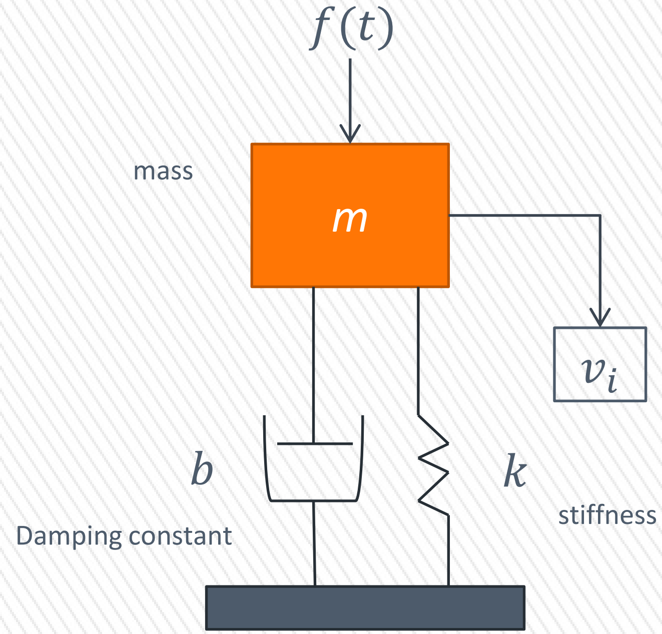


- Note that resistance only has magnitude.

- When a circuit is driven by DC there is no distinction between impedance and resistance (impedance with phase angle to be zero.

TF Electromechanical systems

» *Simple oscillator, single degree of Freedom mass spring damper*



TF: $G(j\omega) = \frac{1}{ms^2+bs+k}$ (1) where $s = j\omega$

$\omega = excitation\ frequency$

$\sqrt{K/m}$ = *system natural frequency*

Scenario 1: When $\omega \ll \sqrt{k/m}$, ms^2 and bs can be neglected with respect to k → system behaves as a simple spring

Scenario 2: when $\omega \gg \sqrt{k/m}$, bs and *k* can be neglected in comparison to ms^2 → *system behaves like a mass element.*

Scenario 3: when

$s = j\omega = j\sqrt{k/m}$ $(i.e.\ excitation\ \omega\ very\ close\ to\ the\ natural\ frequency)$ *then the transfer function G(jω)=1/bs with s=jω*

Properties of Transfer Functions

» In mechanical systems, any type of force or motion variable can be used as the input or output variable of a transfer function.

Transfer Function	In Laplace or Frequency Domain
Dynamic Stiffness	Force/Displacement $= Z \times j\omega$
Receptance (flexibility)	Displacement/force = Mobility/($j\omega$)
Mechanical Impedance Z	Force/velocity
Mobility M	Velocity/force
Dynamic Inertia	Force/acceleration = Impedance/($j\omega$)
Acceleration	Acceleration/force = Mobility $\times j\omega$
Force Transmissibility T_f	Transmitted force/applied force
Motion Transmissibility T_m	Transmitted velocity/applied velocity

Mechanical Transfer Functions

- Across Variables: are measured across an element as the ***difference between the two ends*** *(e.g. velocity, temperature, voltage, pressure)*
- Through Variables: are constant ***properties that flows throughout the element*** *(e.g. force, current, flow, heat transfer rates)*
- Pending the most appropriate representation of the state variable of an element, the element can be *A-type* or *T-type* (for example in Mechanical system mass is A-type and spring is T-type)

Example A-Type : velocity

Example of T-Type : force

(Lecture note 1)

Recall A-Type and T-Type variables

» Once the Transfer Functions of each component are known

» The interconnection laws can be used to determine the overall transfer function of the interconnected system

The two types of interconnections are:

1. Series: The connected elements' through variable is common and the across variables add

2. Parallel: The connected elements' across variables are common and the through variables add.

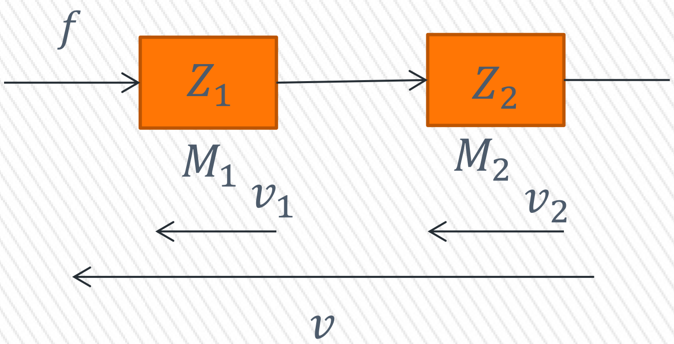
Interconnection Laws

$$Mobility = velocity/force$$
$$Impedance = force/velocity$$

T- Type   A- Type

Interconnection Laws for Mechanical Impedance (*Z*) and Mobility (*M*)

Series



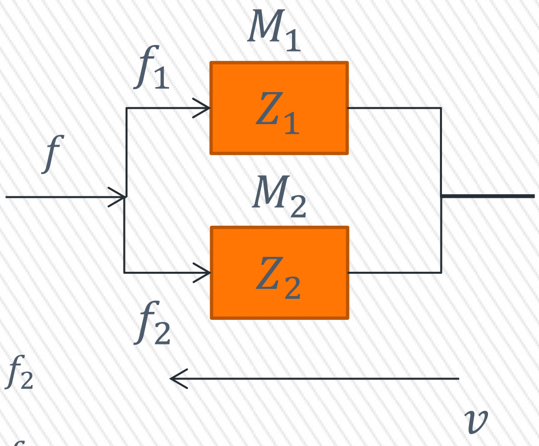
$$v = v_1 + v_2$$

$$\frac{v}{f} = \frac{v_1}{f} + \frac{v_2}{f}$$

$$M = M_1 + M_2$$

$$\frac{1}{Z} = \frac{1}{Z_1} + \frac{1}{Z_2}$$

Parallel



$$f = f_1 + f_2$$

$$\frac{f}{v} = \frac{f_1}{v} + \frac{f_2}{v}$$

$$Z = Z_1 + Z_2$$

$$\frac{1}{M} = \frac{1}{M_1} + \frac{1}{M_2}$$

Interconnection Laws for Mechanical Impedance and Mobility

Interconnection Laws for Electrical Impedance and Admittance

Admittance (W) = Current (i) / Voltage (v)

Series	Parallel
$v = v_1 + v_2$	$i = i_1 + i_2$
$\frac{v}{i} = \frac{v_1}{i} + \frac{v_2}{i}$	$\frac{i}{v} = \frac{i_1}{v} = \frac{i_2}{v}$
$Z = Z_1 + Z_2$	$W = W_1 + W_2$
$\frac{1}{W} = \frac{1}{W_1} + \frac{1}{W_2}$	$\frac{1}{Z} = \frac{1}{Z_1} + \frac{1}{Z_2}$

» The basic (linear) transfer functions for the mass, spring and the damper are given as follows:

Element	Time Domain Model	Impedance	Mobility
Mass (*M*)	$f = m\frac{dv}{dt}$	$Z_m = ms$	$M_m = \frac{1}{ms}$
Spring (*k*)	$\frac{df}{dt} = kv$	$Z_k = \frac{k}{s}$	$M_k = \frac{s}{k}$
Damper (*b*)	$f = kv$	$Z_b = b$	$M_b = \frac{1}{b}$

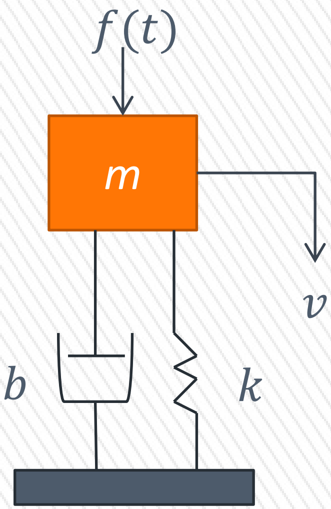
Basic TF for Mechanical Impedance and Mobility

Basic TF for Electrical Impedance and Admittance

» The basic (linear) function for linear electrical elements (circuits):

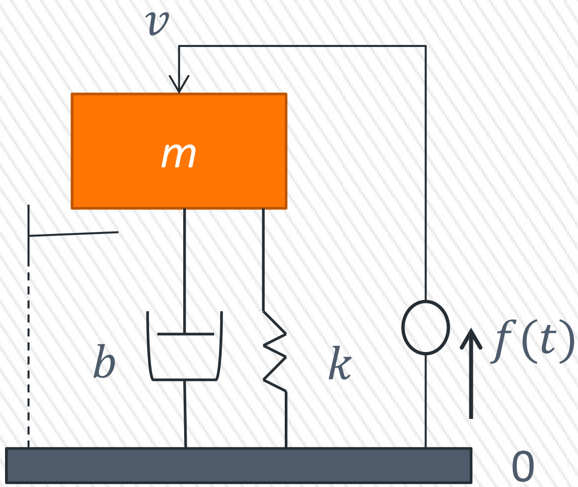
Element	Time Domain Model	Impedance (*Z*)	Admittance (*W*)
Capacitor (*C*)	$C\frac{dv}{dt} = i$	$Z_C = \frac{1}{Cs}$	$W_i = Cs$
Inductor (*L*)	$L\frac{dv}{dt} = v$	$Z_L = Ls$	$W_L = \frac{1}{Ls}$
Resistor (*R*)	$Ri = v$	$Z_R = R$	$W_R = \frac{1}{R}$

» Ground Based Mechanical Oscillator



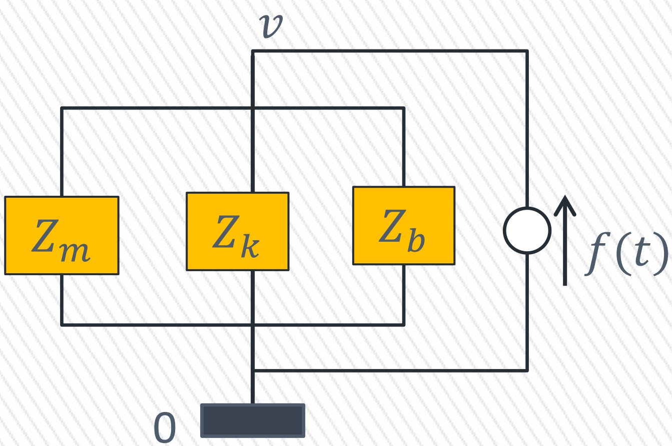
1

Simple Oscillator



2

Mechanical Representation



3

output

$$M = \frac{V(j\omega)}{F(j\omega)}$$

input

output

$$Z = \frac{F(j\omega)}{V(j\omega)}$$

input

Example use of impedance and mobility methods in frequency domain

imagine

1. that a force is applied to such a system where IC=0
2. And we measure the velocity
3. If we move the mass exactly at the same velocity
4. then the force generated will be identical to the original applied force
5. i.e. mobility is the inverse of impedance

The overall impedance function (see figure 3):

$$Z(j\omega)=\frac{F(j\omega)}{V(j\omega)}=Z_m+Z_k+Z_b=ms+\frac{k}{s}+b\,\Big|_{s=j\omega}=\frac{ms^2+bs+k}{s}\,\Big|_{s=j\omega}$$

The mobility function

$$M(j\omega)=\frac{V(j\omega)}{F(j\omega)}=\frac{s}{ms^2+bs+k}\,\Big|_{s=j\omega}$$

Example cont.

imagine

1. that a force is applied to such a system where IC=0
2. And we measure the velocity
3. If we move the mass exactly at the same velocity
4. then the force generated will be identical to the original applied force
5. i.e. mobility is the inverse of impedance

The overall impedance function (see figure 3):

$$Z(j\omega) = \frac{F(j\omega)}{V(j\omega)} = Z_m + Z_k + Z_b = ms + \frac{k}{s} + b \Big|_{s=j\omega} = \frac{ms^2 + bs + k}{s} \Big|_{s=j\omega}$$

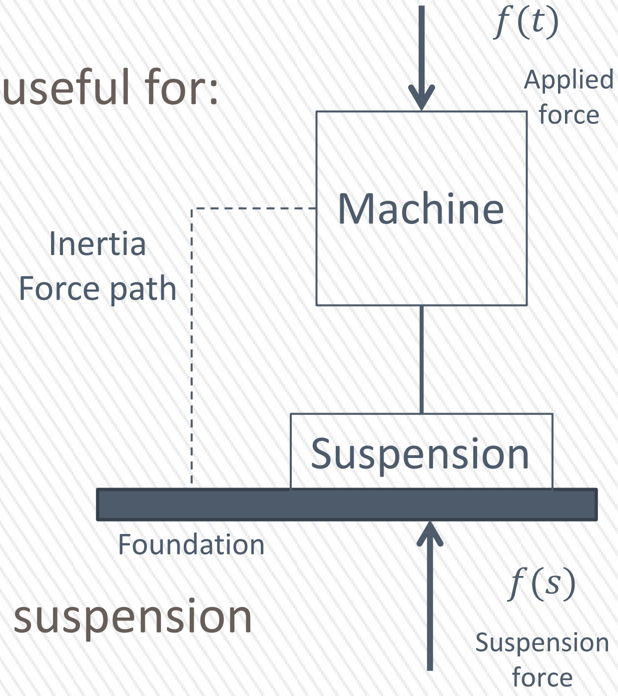
The mobility function

$$M(j\omega) = \frac{V(j\omega)}{F(j\omega)} = \frac{s}{ms^2 + bs + k} \Big|_{s=j\omega}$$

Example cont.

Transmissibility Functions

- Transmissibility functions are transfer functions useful for:
  - Design and analysis of fixtures
  - Mounts
  - Machine support structures
  - Vehicle suspension design and other systems



- Two types of **force** and **motion transmissibility**

- Mass supported on a rigid foundation through a suspension

Force transmissibility $T_f = \frac{F_s}{F}$

Motion Transmissibility $T_m = \frac{V_m}{V} = \frac{system\ motion\ (velocity\ in\ frequency\ domain)}{support\ motion\ (velocity\ in\ frequency\ domain)}$

- Further reading: C. W. de Silva, Modelling and Control of Engineering Systems (2009), Chapter 5 – also read Single Degree of freedom and Two Degree of freedom systems

» The response of a dynamic system can be determined by solving the differential equation (**Analytical**) – subject to ICs

» This can be achieved:

1. Time domain (direct calculations)
2. Laplace Transforms

Consider the following time invariant differential equations:

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_0 y = u \qquad (1)$$

Response Analysis

Homogeneous Solutions

- The characteristics of a dynamic system does not depend on the input to the system
- Therefore the natural behaviour (free response) for equation 1 can be calculated by:

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_0 y = 0 \quad (2)$$

(homogeneous equations)

The solution for linear system (y_h) can be expressed as:

$$y_h = ce^{\lambda t} \quad (3)$$

Where *c* is a constant and λ is complex number.

Now apply equation 3 into 2, knowing that $\frac{d}{dt} e^{\lambda t} = \lambda e^{\lambda t}$

$$a_n \lambda^n + a_{n-1} \lambda^{n-1} + \cdots + a_0 = 0 \quad (4) \quad \text{Characteristic Equation (CE)}$$

» The CE thus has *n* routes : $\lambda_1 \ldots \lambda_n$

» The overall solution to equation $a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_0 y = 0$

Becomes:

$$y_h = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \cdots c_n e^{\lambda_n t}$$

The *c* are determined by the necessary *n* ICs.

Homogeneous Solutions contd.

- Consider a general transfer function: $G(s) = K\frac{N(s)}{D(s)}$

- Where *N(s) and D(s)* are polynomial functions in *(s):*

$$N(s) = s^m + b_{m-1}s^{m-1} + b_{m-2}s^{m-2} + \mathrm{K} + b_1 s + b_0$$

$$D(s) = s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \mathrm{K} + a_1 s + a_0$$

- For causal systems $n \geq m$ i.e. proper system
- Usually $n > m$, *n is the order of the system*
- *D(s)* is the characteristic polynomial (CP)of *G(s)*.
- *D(s)=0* is the is the characteristic equation (CE) of *G(s)*
- The roots of CE are the called the **poles** of the system
- The roots of *N(s)=0* are called the ***zeros*** of *G(s),* therefore:

Properties of TF - Response

$$G(s) = K\frac{N(s)}{D(s)} = \frac{K\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{n}(s+p_j} = \frac{K(s+z_1)(s+z_2)K(s+z_m)}{(s+p_1)(s+p_2)K(s+p_n)}$$

Where z_i and p_i are either real or occur in complex conjugate pairs

The Poles of $G(s)$: $-p_1, -p_2, \mathrm{K}, -p_n$

- Each of the 1st order factors $(s+p_i)$ will give rise to a term $c_i e^{-p_i t}$ in the system transient response.

  $e^{-p_i t}$ is termed as the **mode** of *G(s)*

System transient response

if p_i is complex → $p_i = \sigma_i \pm j\omega$ then $c_i e^{-p_i t} = c_i e^{-\sigma_i t} e^{mj\omega_i t}$

Zeros of $G(s)$: $-z_1, -z_2, K - z_m$

- The zeros influence the magnitude of c_i of each component

Properties of TF continued

Given a situation: $G(s) = \frac{K(s+z)}{(s+p_1)(s+p_2)}$ and the input to be $X(s) = \frac{A}{s}$

Let $p_1 \neq p_2$ then:

$$Y(s) = \frac{KA(s+z)}{s(s+p_1)(s+p_2)}$$

$$= KA\left\{\frac{z}{p_1 p_2}\frac{1}{s} + \frac{(1-z/p_1)}{(p_2-p_1)}\frac{1}{s+p_1} + \frac{(1-z/p_2)}{(p_1-p_2)}\frac{1}{s+p_2}\right\}$$

Or

$$y(t) = KA\left\{\frac{z}{p_1 p_2} H(t) + \frac{(1-z/p_1)}{(p_2-p_2)} e^{-p_1 t} + \frac{(1-z/p_2)}{(p_1-p_2)} e^{-p_2 t}\right\}$$

As $z \rightarrow p_1$, the magnitude $c_1 \rightarrow 0$ As $z \rightarrow p_2$, the magnitude $c_2 \rightarrow 0$

When $z = p_1 \;\; or \; z = p_2,$ then **pole zero** condition and **cancellation in the *G(s)*** occurs

A mode in the response is supressed (i.e. $G(s) = \frac{K(s+p_1)}{(s+p_1)(s+p_2)} = \frac{K}{s+p_2} \;\; or \;\; \frac{K}{s+p_1}$

Example

Once pole-zero cancellations occur all the cancellations on *G(s)* are made we have: the ***minimal realisation*** of *G(s).*

This is the minimum order representation of G(s).

In our example $\frac{K}{s+p_2}$ and $\frac{K}{s+p_1}$ are the minimal realisations

Example Contd.

Pole-Zero Plot and the *s - plane*

- The poles and zeros of *G(s)* are given by particular values for *s.*

- *s* is a complex variable expressed as:

in the cartesian form $s = \sigma + j\omega$

In polar form $re^{j\theta}$ where $r = \sqrt{\sigma^2 + \omega^2}$

and $\theta = tan^{-1}(^{\omega}/_{\sigma})$

A graphical representation of transient response is given by a plot of all pole and zero positions on a complex plane called the ***s* - plane**.

- poles shown by ‘x’
- zeros shown by ‘o’

Example

» Find the system order, characteristic polynomial, characteristic equation, the system poles, the system zeros, the modes and draw the pole/zero plots for the following systems:

Case 1 $$G(s) = \frac{3}{s^2+4s+3}$$

System order = order of the denominator polynomial = 2

CP: $s^2 + 4s + 3$ or *(s+1)(s+3)*

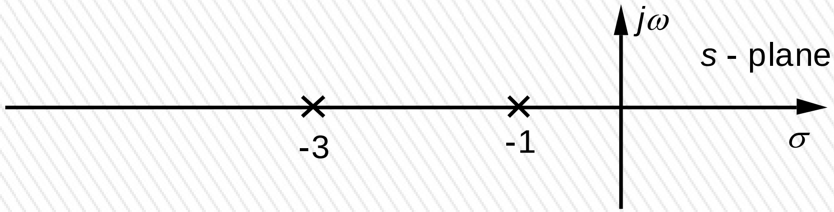
CE: $s^2 + 4s + 3$=0 or *(s+1)(s+3)=0*

Poles: (solution to CE, thus s= -1 , -3

No zeros

Modes: e^{-t}, e^{-3t}

The pole/zero plot looks like:



Case 2: $G(s) = \frac{2(s+2)}{s^3+5s^2+11s+15} = \frac{2(s+2)}{(s+3)(s^2+2s+5)}$

system order = order of denominator polynomial = 3

CP: $s^3 + 5s^2 + 11s + 15 = (s+3)(s^2+2s+5)$

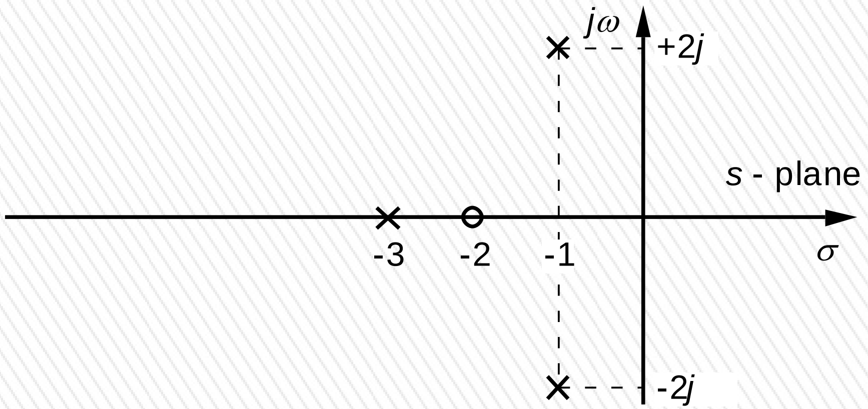
CE: $s^3 + 5s^2 + 11s + 15 = 0$ or $(s+3)(s^2+2s+5) = 0$

poles: (solution to CE) $s = -3, -1 \pm j2$

zeros: *s = -2*

Modes: $e^{-3t}, e^{-(1 \pm j2)}$

The pole/zero plot:



Example Contd.

» The relationship between the input and output of a first order differential equation:

$$a_1 \frac{dx}{dt} + a_0 x = b_0$$

with the Laplace transform:

$$a_1 X(s) + a_0 X(s) = b_0 Y(s)$$

And the transfer function:

$$G(s) = \frac{X(s)}{Y(s)} = \frac{b_0}{a_1 s + a_0}$$

$$\therefore G(s) = \frac{b_0/a_0}{\left(\frac{a_1}{a_0}\right) s + 1} = \frac{G}{\tau s + 1}$$

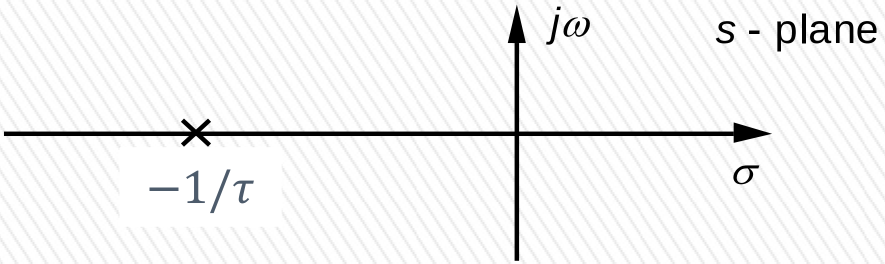
Gain (scales the response)

Time constant

First Order Systems

First Order Systems contd.

- For simplicity let *G=1*
- *The pole-zero plot:*



- When a unit step input is applied to a first order system, then $Y(s) = {}^{1}/_{s}$
- The output will then be:

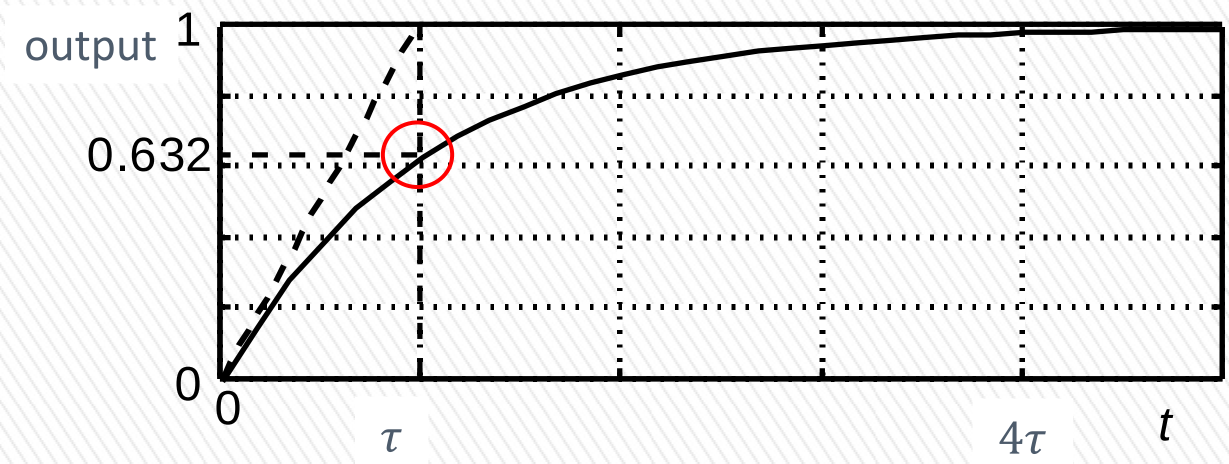
$$X(s) = G(s)Y(s) = \frac{G}{s(\tau s + 1)} = G\frac{1/\tau}{s(s + \frac{1}{\tau})}$$

By referring to the transform table : $\frac{a}{s(s=a)} \rightarrow (1 - e^{-at})$

Then the unit step response: $x = G\left(1 - e^{-\frac{t}{\tau}}\right)$

» $x(t) = \left(1 - e^{-\frac{t}{\tau}}\right)H(t)$

» The unit response looks like:



» the commonly used measure of the speed of response is the time constant:

when $t = \tau$, the exponential has decayed to $e^{-1} = 0.368$, of its initial value

In other words the step response has reached 1-0.368=0.632 of its final value

First Order Systems contd.

For a simple 1st order system , the time constant is τ

writing the transfer function in this form means T can be seen directly

From the pole-zero plot,

- » the system pole $-{}^{1}/_{\tau}$ must lie on the negative real axis for stability
- » the further to the left the system pole lies, the faster the response (the time constant is decreased)

First Order Systems contd.

» A simple example to illustrate the behaviour of the transfer function of a first order system when subject to a step input

Consider a circuit which consists of a resistor (R) and a capacitor (C) in series. The input is v and the output is the potential difference v_c across the capacitor, the differential equation:

$$v = RC\frac{dvc}{dt} + vc$$

When IC=0

The Laplace transform: $V(s) = RC_sV_c(s) + V_c(s)$

Transfer function: $G(s) = \frac{V_C(s)}{V(s)} = \frac{1}{RCs+1}$

Example

Consider a thermocouple with a transfer function linking voltage output V and temperature input:

$$G(s) = \frac{30 \times 10^{-6}}{10s + 1}$$

What is the response of the system when a step input of 120°C and the time to reach 95% of the steady state value?

The transform of the output = transfer function X transform of the input

$$V(s) = G(s) \times input(s)$$

The temperature abruptly increased by 120°C, is 100/s

$$V(s) = \frac{30 \times 10^{-6}}{10s + 1} \times \frac{100}{s} = 30 \times 10^{-4} \frac{0.1}{s(s + 0.1)}$$

$$\frac{a}{s(s + a)}$$

The inverse transform: $V = 30 \times 10^{-4}(1 \times e^{-0.1t})\text{V} \,\Big|_{t \to \infty}$ →

Example

When $t \rightarrow \infty$ the exponential value = 0, the final value therefore:

$$\therefore 30 \times 10^{-4}$$

Therefore the time to reach the 95% of this is given by;

$$0.95 \times 30 \times 10^{-4} = 30 \times 10^{-4}(1 \times e^{-0.1t})$$

So $0.05 = e^{-0.1t}$ and $\ln 0.05 = -0.1t$ therefore *t = 0.5 min.*

Example contd.