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Embedded Systems and Industrial Controller EE5563 (5)

Topics

- 2nd order systems
- General characterisation of 2nd TF
- Stability of System
- Routh-Hurwitz Stability Criterion

2^{nd} Order Systems

» The differential equation of a 2^{nd} order system:

$$a_2 \frac{d^2y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = b_0 u \qquad (1)$$

The Laplace transform of a simple 2^{nd} order system with IC=0:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_0}{a_2 s^2 + a_1 s + a_0} \qquad (2)$$

» Consider:

$$\mathrm{G}(s) = \frac{6}{s^2 + 5s + 6}$$

Apply a unit step $H(t) = \frac{1}{u(t)} \Rightarrow U(s) = \frac{1}{s}$

Therefore;

$$Y(s) = G(s)U(s) \Rightarrow \frac{6}{s(s^2 + 5s + 6)} = \frac{6}{s(s+2)(s+3)}$$

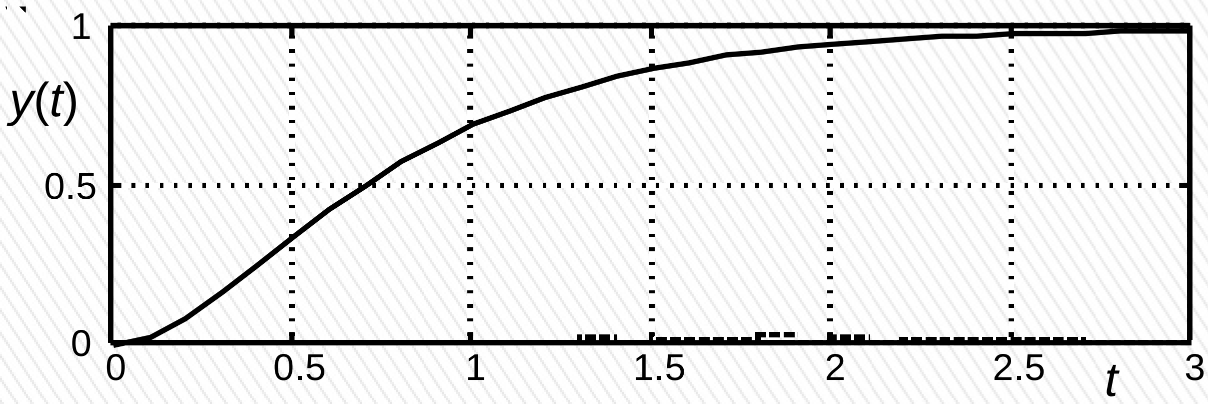
Expanding the *X(s)* into partial fractions:

$$Y(s) = \frac{1}{s} - \frac{3}{s+2} + \frac{2}{s+3}$$

From the standard table of Laplace Transforms:

$$y(t) = (1 - 3e^{-2t} + 2e^{-3t})H(t)$$

The unit step:



2nd Order Systems contd.

» Consider a system with TF: $G(s) = \frac{10}{s^2+2s+5}$

» Apply a unit step input:

$$Y(s) = \frac{10}{s\left(s^2+2s+5\right)} = \frac{2}{s} - \frac{2s+4}{s^2+2s+5}$$

$$\rightarrow \quad Y(s) = \frac{2}{s} - \frac{2(s+1)+2}{(s+1)^2+(5-1)} = \frac{2}{s} - \frac{2(s+1)}{(s+1)^2+2^2} - \frac{2}{(s+1)^2+2^2}$$

So $y(t) = (2 - 2e^{-t}cos2t - e^{-t}sin2t)H(t)$

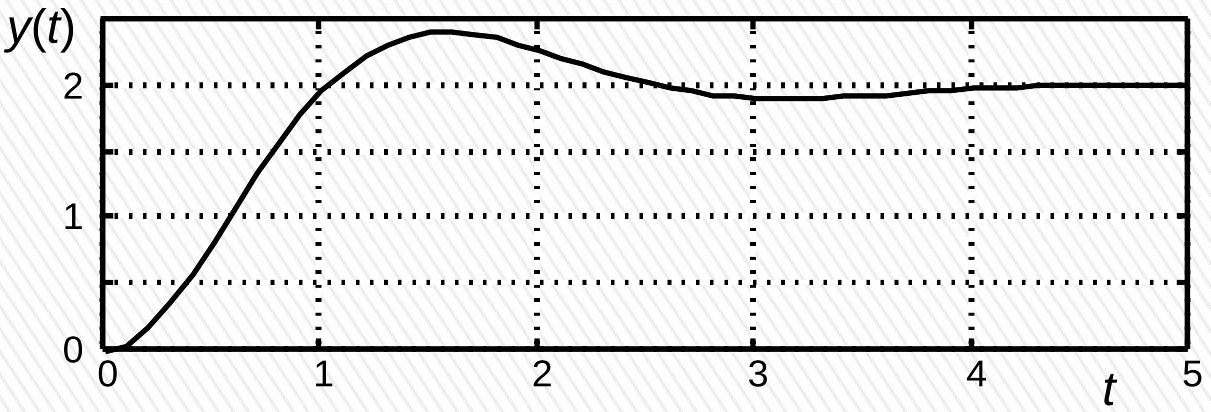
It can be converted into: $y(t) = 2 - Ae^{-t}\sin(2t + \varphi)$

Where: $A = \sqrt{2^2 + 1^2} = \sqrt{5}$

And $\varphi = tan^{-1}(^2/_1) = 1.11\, rad\ [63.4°]$

2nd Order Systems contd.

The unit step response will look like:



2nd Order Systems contd.

$$G(s) = \frac{\omega^2{}_n}{s^2+2\zeta\omega_n s+\omega^2{}_n} \qquad (3)$$

Where ζ is the *damping ratio*

And

ω_n is the *natural angular (undamped) frequency*

Where four scenarios:

» *ζ > 1 overdamped (real, distinct factors)*

» *ζ = 1 critically damped (real, repeated factors)*

» *ζ < 1 underdamped (complex factors)*

The factors occur as a complex conjugate pair

» *ζ = 0 undamped*

The general representation of 2nd Order TF

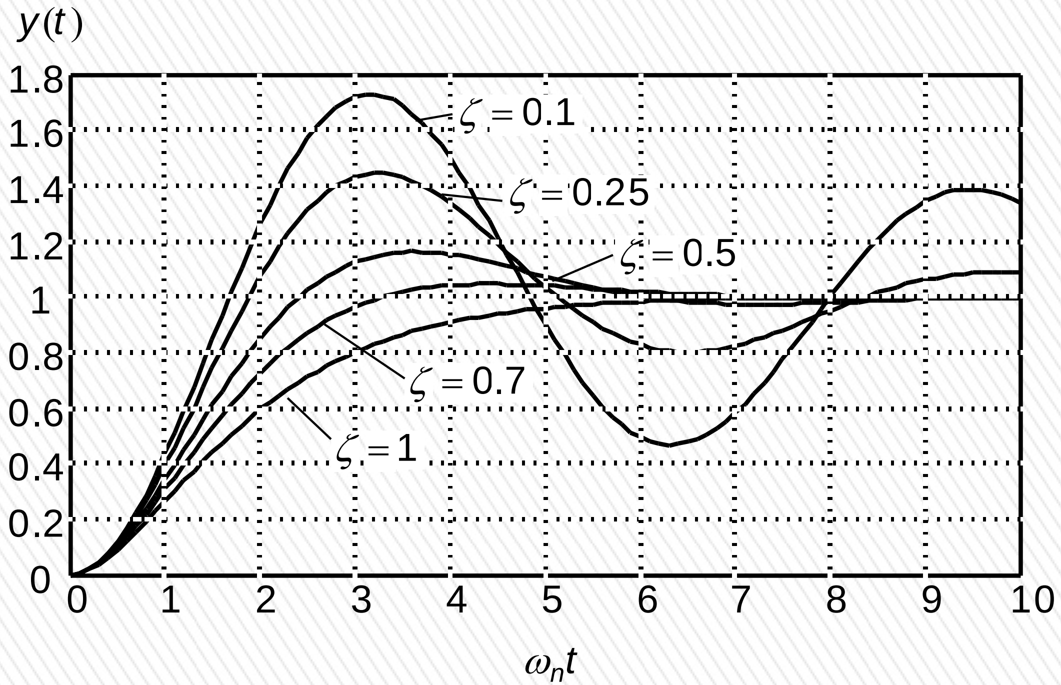
For the underdamped case (0 < ζ <1), the unit step response with zero IC would be:

$$y(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_n t\sqrt{1-\zeta^2} + cos^{-1}\zeta) \quad (4)$$

For different ζ values the unit step response would look like:

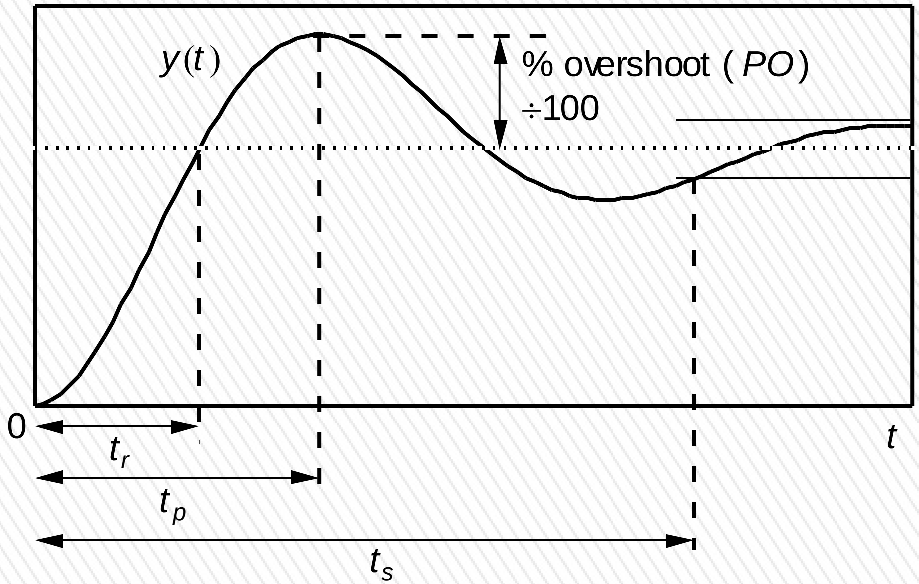
The frequency of oscillation is:

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$



The general representation of 2nd Order TF cont.

It helps to characterise the 2nd order step response as:



Where:

t_r is the *rise time*

t_p is the *first peak*

PO is the *% overshoot*

t_s is the *setting time* for the response to fall within a prescribed band round about the final steady state value

General Characterisation of the 2nd order step response

t_p can be found by differentiating *y(t)* and setting the derivative equal to zero.

The time for the first peak is: $t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$

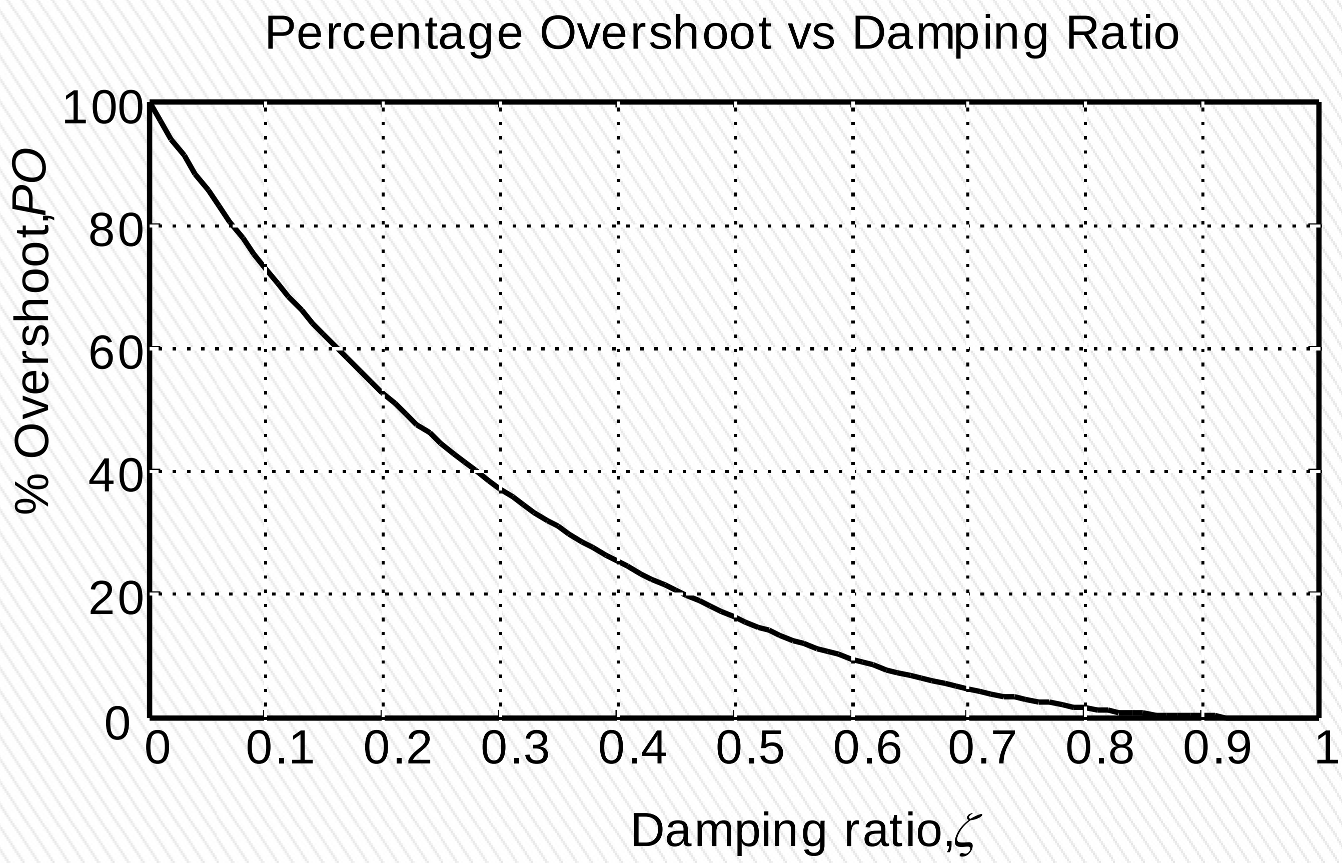
Alternatively, note that the transient response oscillates at the *damped natural frequency*, $\omega_d = \omega_n\sqrt{1-\zeta^2}$

→ Time to first peak = ½ of the period = $\frac{1}{2} \times \frac{2\pi}{\omega_d} = \frac{\pi}{\omega_d}$

At this point the output will be: $y(t_p) = 1 + exp\left(\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right)$

$$\therefore PO = exp\left(\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right) \times 100\%$$

General Characterisation of the 2nd order step response cont.



% overshoot against damping ratio

General Characterisation of the 2nd order step response cont.

t_r is found by setting *y(t) = 1* an solving the lowest value of *t*

Giving $\omega_n\sqrt{1-\zeta^2}t_r + cos^{-1}\zeta = \pi$

$$\therefore t_r = \frac{\pi - cos^{-1}\zeta}{\omega_n\sqrt{1-\zeta^2}}$$

t_s is found by the amplitude term $e^{-\zeta\omega_n t}$ in *y(t)* *(see equation 4 slide 8)*

» this is a decaying exponential term with time constant of $\frac{1}{\zeta\omega_n}$

The response would decay within 5% of the final value after 3 times constants or 2% of the final value after 4 times constants etc.

So for settling to within 5%, $t_s = \frac{3}{\zeta\omega_n}$ and 2%, $t_s = \frac{4}{\zeta\omega_n}$

» A system can be categorised as ***stable*** if it is excited by an input, it has transients which dissipate in time and leave the system in its steady-state condition.

» A system can be considered as ***unstable*** if the transients do not die off with time but increase in size, thus the steady-state is never reached.

» Consider an input of a unit of impulse to a first order system with:

$G(s) = \frac{1}{s+1}$ the system output *Y(s):*

$$Y(s) = G(s)U(s) = \frac{1}{s+1} \times 1$$

and thus the $y = e^{-t}$ as time *t* increases the output dies away equal to zero.

Stable System

Now if the transfer function is: $G(s) = \frac{1}{s-1}$, $Y(s) = \frac{1}{s-1} \times 1$.

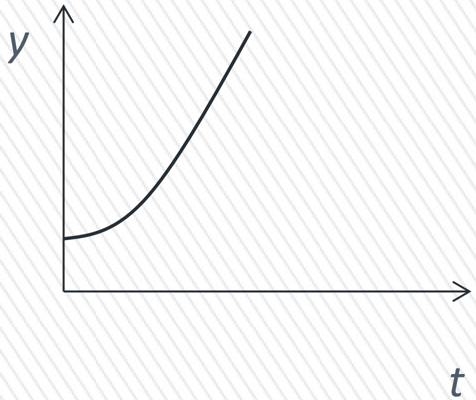
Therefore, $y = e^{t}$, as *t* increases output increases, ever increasing output. Thus the system is **Instable.**

Effect of pole location on transient response

Poles of TF

- Remember that the values of ***s*** that make the TF infinite are the **Poles.**
- In other words the roots of characteristic equation.

In the case of $G(s) = \frac{1}{s-1}$, *s = 1 and* $G(s) = \frac{1}{s+1}$ *, s = -1 are the poles.*



unstable (s positive)

stable (s negative)

Poles for second order system

- For 2nd order system with TF:

$$G(s) = \frac{\omega^2{}_n}{s^2 + 2\zeta\omega_n s + \omega^2{}_n}$$

When subject to a unit of impulse input:

$$Y(s) = \frac{\omega^2{}_n}{(s+p_1)(s+p_2)}$$

Where p_1 and p_2 are the roots of equation:

$$s^2 + 2\zeta\omega_n s + \omega_n = 0$$

Using the equation for roots of quadratic equation:

$$p = \frac{-2\zeta\omega_n \pm \sqrt{4\zeta^2\omega^2{}_n - 4\omega^2{}_n}}{2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

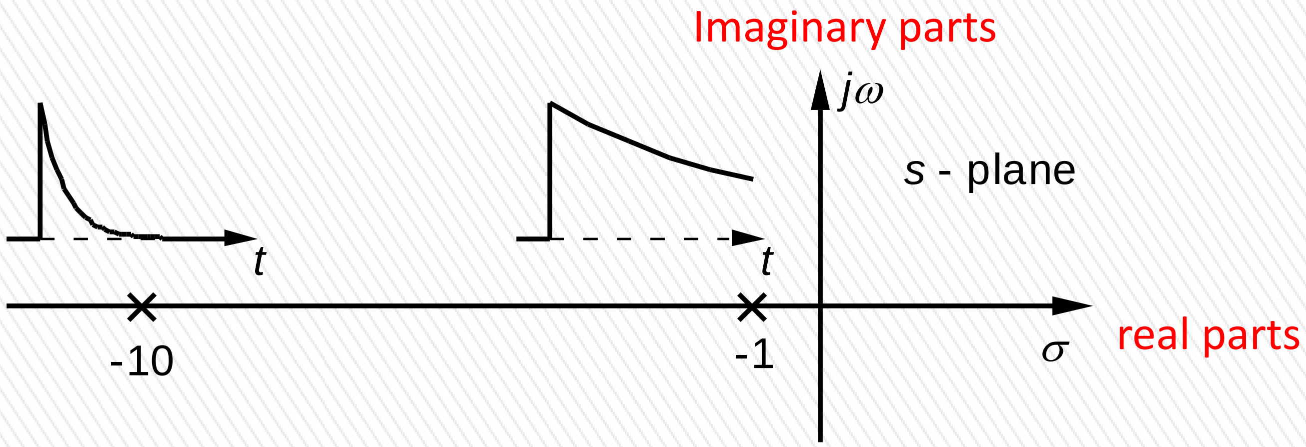
» Depending on the value of ζ (damping factor), the term under the square root can be real or imaginary.

» In case there is an imaginary term the output involves an oscillation.

Let’s look at 2 scenarios

Poles for second order system cont.

1. $G(s) = \frac{K}{s^2+11s+10} = \frac{K}{(s+1)(s+10)}$ two real poles : **stable**



System mode will persist much longer due to pole at *s = -1* than the system mode due to pole at *s = -10*.

No zero to influence the magnitude of the modes.

Therefore pole at *s = -1* will *dominate* transient response of system

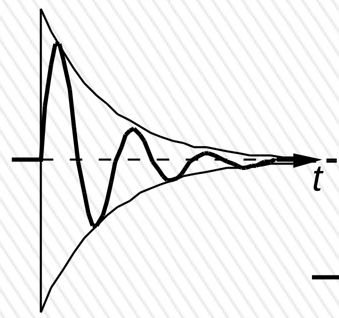
Pole/Zero Plots for 2nd order system

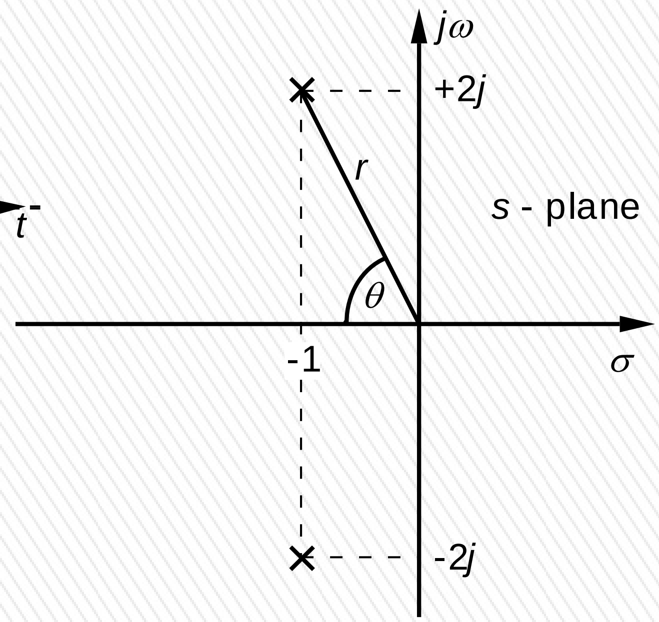
Pole/Zero Plots for 2nd order system cont.

2. $G(s) = \frac{K}{s^2+2s+5} = \frac{K}{(s+1+2j)(s+1-2j)}$ two complex conjugate poles

$s^2 + 2\zeta\omega_n s + \omega^2{}_n$ $\omega_n = \sqrt{5}$; $\zeta = \frac{1}{\sqrt{5}}$

$s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$ stable system





$$\left.\begin{aligned} \omega_n = r = \sqrt{2^2 + 1^2} = \sqrt{5} \\ \zeta = cos\theta = \frac{1}{\sqrt{5}} \end{aligned}\right\}$$ confirm from the characteristic polynomial

The frequency of oscillation, ω_d in the output response

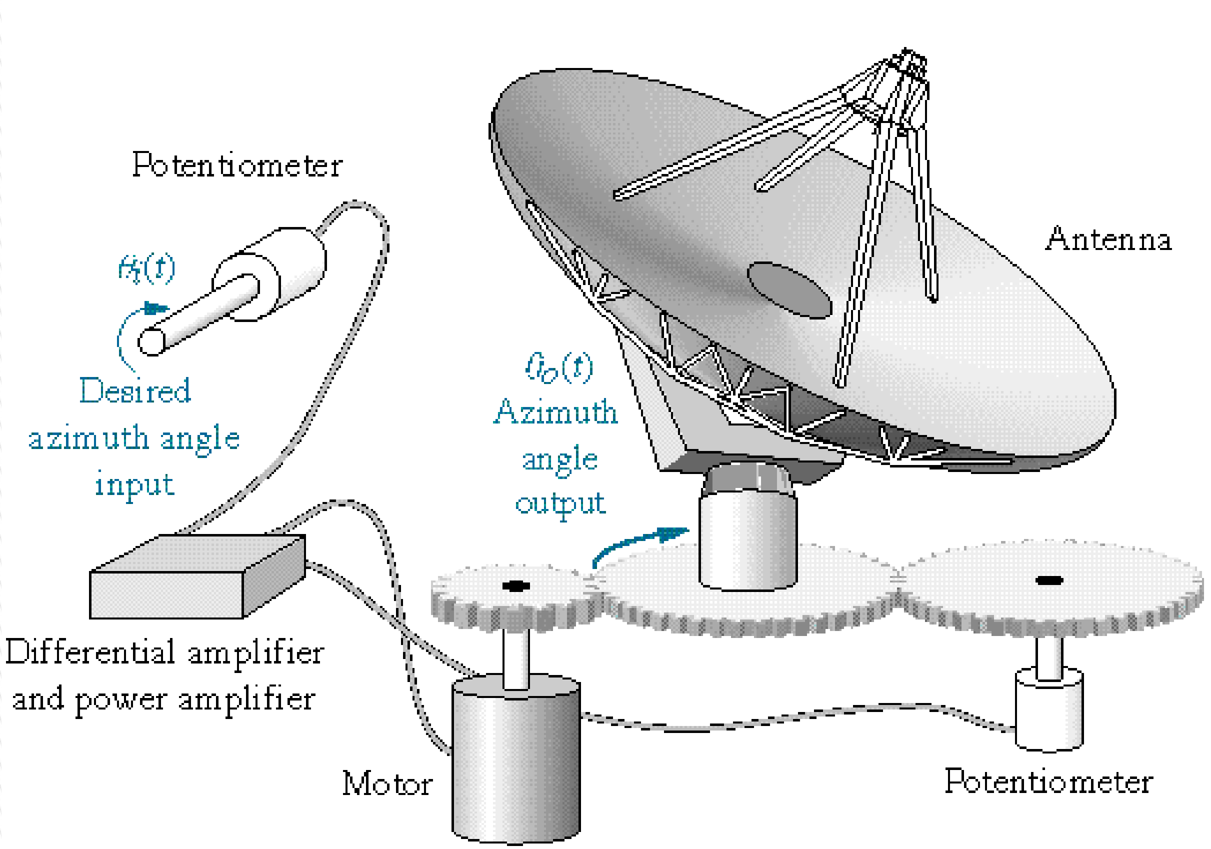
- given by the imaginary component of the poles ($\omega_d = 2rads^{-1}$ in this case)

- Conforming that $\omega_d = \omega_n\sqrt{1 - \zeta^2}$

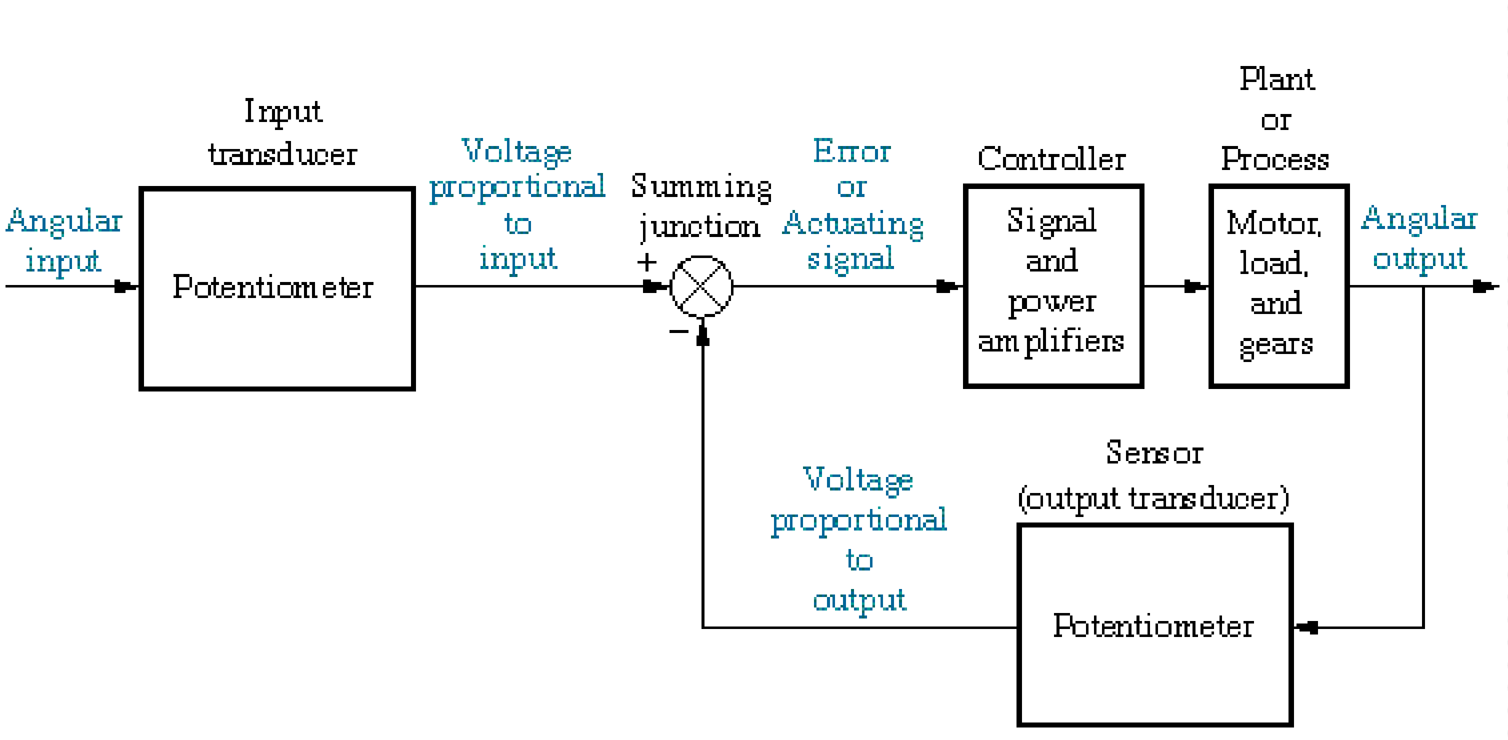
two complex conjugate poles

Antenna Azimuth Position Control System

The diagram of the system:



Case Study



Case Study The schematic diagram

Stability

» Consider a system described by a 1st order LDE:

$$\frac{dy}{dt} + ay(t) = bu(t)$$

Adopting the Laplace transforms:

$$sY(s) - y(0) + aY(s) = bU(s) \Rightarrow Y(s) = \frac{1}{s+a}y(0) + \frac{b}{s+a}U(s)$$

For stability, we consider the zero input response $y_0(t)$:

$$y_0(t) = y(0)e^{-at}H(t)$$

If $a > 0$ then the system is stable (the exponential term decays)

If $a < 0$ then the system is unstable (the exponential term increases)

If $a = 0$ the exponential term become 1, therefore the response is constant → the system is *marginally* stable

Generally;

» a system is **asymptotically stable** if $y_0(t) \rightarrow 0$ as $t \rightarrow \infty$

i.e. the output response due to initial energy in the system decays to zero as *t* becomes large.

» a system is **marginally stable** if for each set of initial conditions there is a finite positive constant *M* such that:

$|y_0(t)| \leq M$ for all $t \geq 0$

{in general, *M* depends on the initial conditions}

i.e. the output response is bounded in magnitude.

» a system is **unstable** if there are values of the initial conditions for which

$|y_0(t)| \rightarrow \infty$ as $t \rightarrow \infty$

i.e. the output response grows without bound for some values of initial conditions.

Case study cont.

Stability for higher order systems

For higher order systems:

» the free response is determined by the modes of the systems:

$$y_0(t) = C_1 e^{-p_1 t} + C_2 e^{-p_2 t} + \cdots + C_n e^{p_n t}$$

where are the system poles (real or complex)

if a pole is real, the exponential term decays if the pole is negative.

» for complex conjugate poles; $s = \zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$

The response is oscillatory with an exponential term $e^{\zeta\omega_n t}$

> if $\zeta > 0$ (real part of the pole is negative), exponential term decays ⇒ system is stable
>
> if $\zeta < 0$ (real part of the pole is positive), exponential term grows unbounded ⇒ system is unstable
>
> if $\zeta = 0$ exponential term = 1 ⇒ system is marginally stable - constant oscillations

All higher order systems consist of combinations of the above terms

» if just a single exponential term grows unbounded ⇒ system is unstable

Stability Theorem:

A system is stable if and only if all system poles lie in the left half of the *s*-plane.

Stability for higher order systems cont.

» The Routh-Hurwitz stability criterion is simple way to determine whether a system is stable or not (i.e. whether non of the poles have positive real parts).

» It is achieved by examining the characteristic of polynomial without solving the characteristic equation for its roots.

» If the system is unstable the Routh-Hurwitz test will tell us the number of poles that are on the right half of the s-plane

» First step is to establish the Routh array.

Routh-Hurwitz Stability Criterion

The Characteristic Equation (CE) can be expressed as:

$$a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \cdots + a_0 = 0$$

- » In order for the system to be stable, all the roots must lie on the left half of the s-plane.
- » The necessary condition for stability is that all the coefficients are present and have the same sign.
- » Sufficient conditions are obtained from the following table:

s^n	a_n	a_{n-2}	a_{n-4}	…	1st row
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	…	2nd row
s^{n-2}	b_1	b_2	b_3	…	3rd row
s^{n-3}	c_1	c_2	c_3	…	4th row
…	.	.	.	…	
s^0	h_1				Last row

Routh-Hurwitz Stability Criterion Cont.

» The first two rows are completed using the $a_n, a_{n-1}, \dots, a_1, a_0$ (coefficients)of the characteristic polynomial.

» Note the alternate coefficients in these two first rows. i.e. first row $[a_n, a_{n-2}, \dots]$ and second row $[a_{n-1}, a_{n-3}, \dots]$

» Each subsequent row is calculated from elements of the two immediately above rows, by cross-multiplying the elements of the two rows:

$$b_1 = \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}},$$

$$b_2 = \frac{a_{n-1}a_{n-4} - a_n a_{n-5}}{a_{n-1}}, \quad \text{or } b_{n-k} = -\frac{1}{a_{n-1}}\begin{bmatrix} a_n & a_{n-k} \\ a_{n-1} & a_{(2k+1)} \end{bmatrix}$$

$$c_1 = \frac{b_1 a_{n-3} - a_{n-1} b_2}{b_1},$$

$$c_2 = \frac{b_1 a_{n-5} - a_{n-1} b_3}{b_1}, \text{ or } c_{n-k} = -\frac{1}{b_{n-1}}\begin{bmatrix} a_{n-1} & a_{n-(2k+1)} \\ b_{n-1} & b_{n-(k+1)} \end{bmatrix}$$

And so on …

Routh-Hurwitz Stability Criterion Cont.

Just to recap again, conditions for stability based on Routh-Hurwitz Criterion:

1. All the coefficients $(a_n, a_{n-1}, \dots, a_n)$ of the characteristic equation should be of same sign (positive or negative).

2. All the elements of the first column of the Routh array must be positive or the same sign.

3. If the system is unstable, the number of unstable poles is given by the number of successive sign changes in the elements of the first column of the Routh array.

The Conditions for Stability based on Routh-Hurwitz Criterion

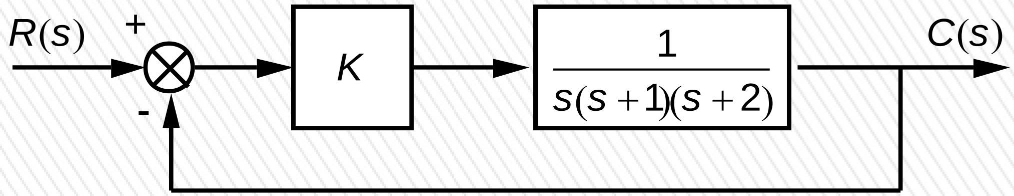
Example 1: Consider a closed-loop system with transfer function to be:

$$G(s) = \frac{1}{(s^3 - s^2 + 2s + 1)}$$

Without creating the Routh array we can see that the system is unstable since condition 1 is not met. Because the coefficients of the CE are not of same sign.

Example 1

Example 2: determine if the following system is stable:



Find the limits on *K* for stability for the following system.

The closed-loop Characteristic equation is $1 + G(s) = 0$

To say $s^3 + 3s^2 + 2s + K = 0$

The Routh array:

$$
\begin{array}{llll}
s^3 & 1 & 2 & 0 \\
s^2 & 3 & K & 0 \\
s^1 & \frac{6-k}{3} & 0 & \\
s^0 & K & 0 &
\end{array}
$$

For all the elements of the 1st column to be positive, we require $0 < K < 6$

Example 2

» Note: at either extreme value, the Routh array has a zero row - for this example this represents roots on the imaginary axis.

» These roots can be found from the auxiliary equation formulated from the elements in the row that precedes the row of zeros.

» For example $K = 6$ the elements of s^1 row are zero. The auxiliary equation is:

$3s^2 + 6 = 0$ which gives the roots $s = \pm j\sqrt{2}$.

Example 2 cont.