Tests for Stability:

• Jury's test

This is an algebraic test, similar in form to the Routh - Hurwitz approach, that determines whether the roots of a polynomial lie within the unit circle.

As for Routh - Hurwitz, the test consists of two parts:

(1) simple test for necessary conditions

(2) test for sufficient conditions

For a polynomial of the form:

$$F(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0 \quad (a_n > 0)$$

the *necessary* conditions for stability are:

$$F(1) > 0$$

and $$(-1)^n F(-1) > 0$$

The *sufficient* conditions for stability are obtained by forming a table as follows:

row	z^0	z^1	z^2	K	z^{n-k}	K	z^{n-1}	z^n
1	a_0	a_1	a_2	K	a_{n-k}	K	a_{n-1}	a_n
2	a_n	a_{n-1}	a_{n-2}	K	a_k	K	a_1	a_0
3	b_0	b_1	b_2	K	b_{n-k}	K	b_{n-1}	
4	b_{n-1}	b_{n-2}	b_{n-3}	K	b_k	K	b_0	
5	c_0	c_1	c_2	K	K	c_{n-2}		
6	c_{n-2}	c_{n-3}	c_{n-4}	K	K	c_0		
M	M	M	M	M				
$2n-5$	p_0	p_1	p_2	p_3				
$2n-4$	p_3	p_2	p_1	p_0				
$2n-3$	q_0	q_1	q_2					

where:

$$b_k = \begin{vmatrix} a_0 & a_{n-k} \\ a_n & a_k \end{vmatrix} \qquad c_k = \begin{vmatrix} b_0 & b_{n-1-k} \\ b_{n-1} & b_k \end{vmatrix} \qquad d_k = \begin{vmatrix} c_0 & c_{n-2-k} \\ c_{n-2} & c_k \end{vmatrix}$$

The sufficient conditions for stability are given by:

$$\left.\begin{array}{l} |a_0| < a_n \\ |b_0| > |b_{n-1}| \\ |c_0| > |c_{n-2}| \\ \vdots \quad\quad \vdots \\ |p_0| > |p_3| \\ |q_0| > |q_2| \end{array}\right\} (n-1) \text{ conditions}$$

These inequality conditions *must* provide conclusive results - singularities occur if the first and last elements of any row are zero.

Singularities can be dealt with by considering an infinitesimal contraction and expansion of the unit circle using the transformation

$$z = (1 + \varepsilon) z$$

where ε is a very small number.

The difference between the no. of roots found inside (or outside) the unit circle when the circle is expanded and contracted by ε is the no. of roots on the unit circle.

The transformation is applied by using:

$$(1 \pm \varepsilon)^n z^n \approx (1 \pm n\varepsilon) z^n$$

$\therefore$ the coefficient of the z^n term is multiplied by $(1 \pm n\varepsilon)$.

Example: CE: $z^2 - z + 2 = 0$

Necessary conditions:

$F(1) = 1^2 - 1 + 2 = 2 > 0$ √

$(-1)^n F(-1) = (-1)^2 \left[(-1)^2 - (-1) + 2 \right] = 4 > 0$ √

Sufficient conditions:

row

1: $a_0 = 2$ $a_1 = -1$ $a_2 = 1$

2: $a_2 = 1$ $a_1 = -1$ $a_0 = 2$

$|a_0| = |2| = 2$ and $a_2 = 1$

$\therefore$ $|a_0| \not< a_2$ system is UNSTABLE

Tests for Stability:

• bilinear transform & Routh - Hurwitz test

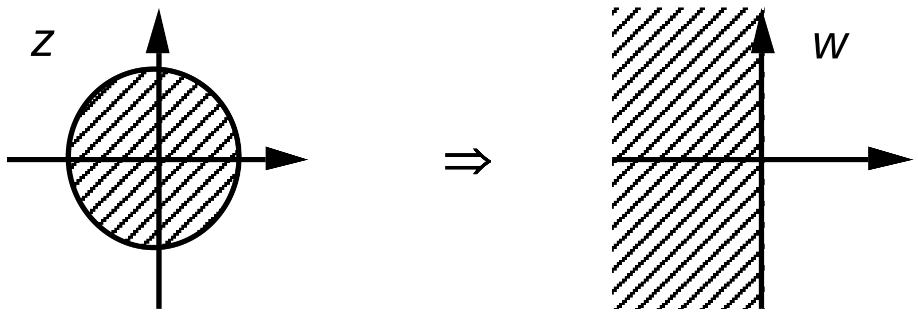
bilinear transform

$$z = \frac{1+w}{1-w} \qquad \left(\text{gives } w = \frac{z-1}{z+1}, \text{ undefined at } z = -1\right)$$

or

$$z = \frac{w+1}{w-1} \qquad \left(\text{gives } w = \frac{z+1}{z-1}, \text{ undefined at } z = 1\right)$$

Maps the inside of the unit circle in the *z* - plane into the LH *w* - plane.



Now can use Routh-Hurwitz criterion on the CE in the *w* - plane.

Review of Routh - Hurwitz

Consists of (1) test for necessary conditions
(2) test for sufficient conditions

For a polynomial of the form

$$F(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0 = 0$$

the necessary condition for stability is that all the coefficients are present and have the same sign.

Sufficient conditions are obtained from the following table:

s^n	a_n	a_{n-2}	a_{n-4}	a_{n-6}	$\cdots$
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	$\cdots$	
s^{n-2}	b_{n-1}	b_{n-2}	b_{n-3}	$\cdots$	
s^{n-3}	c_{n-1}	c_{n-2}	c_{n-3}	$\cdots$	
s^{n-4}	d_{n-1}	d_{n-2}			
$\vdots$	$\vdots$				
s^0					

where

$$b_{n-k} = -\frac{1}{a_{n-1}}\begin{vmatrix} a_n & a_{n-2k} \\ a_{n-1} & a_{n-(2k+1)} \end{vmatrix} \qquad c_{n-k} = -\frac{1}{b_{n-1}}\begin{vmatrix} a_{n-1} & a_{n-(2k+1)} \\ b_{n-1} & b_{n-(k+1)} \end{vmatrix}$$

$$d_{n-k} = -\frac{1}{c_{n-1}}\begin{vmatrix} b_{n-1} & b_{n-(k+1)} \\ c_{n-1} & c_{n-(k+1)} \end{vmatrix}$$

Every change of sign in the first column of this array signifies the presence of a root with a positive real part.

Two different types of singularity can occur:

(a) zero in first column - solution is to let $\sigma = 1/s$ and repeat the procedure.

(b) full row of zeros (indicates diametrically opposite roots) - solution is to solve the auxiliary equation (i.e. the polynomial whose coefficients are the elements of the row immediately above the row of zeros) to give the offending roots. The Routh array is completed by replacing the row by the coefficients of the first derivative of the auxiliary equation.

Example: CLCE: $z^2 - z + 2 = 0$

use $z = \frac{1+w}{1-w}$

$\therefore$ CE becomes $\frac{(1+w)^2}{(1-w)^2} - \frac{(1+w)}{(1-w)} + 2 = 0$

$\therefore$ $2w^2 - w + 1 = 0$

Necessary conditions:

all coefficients present √

and have the same sign x

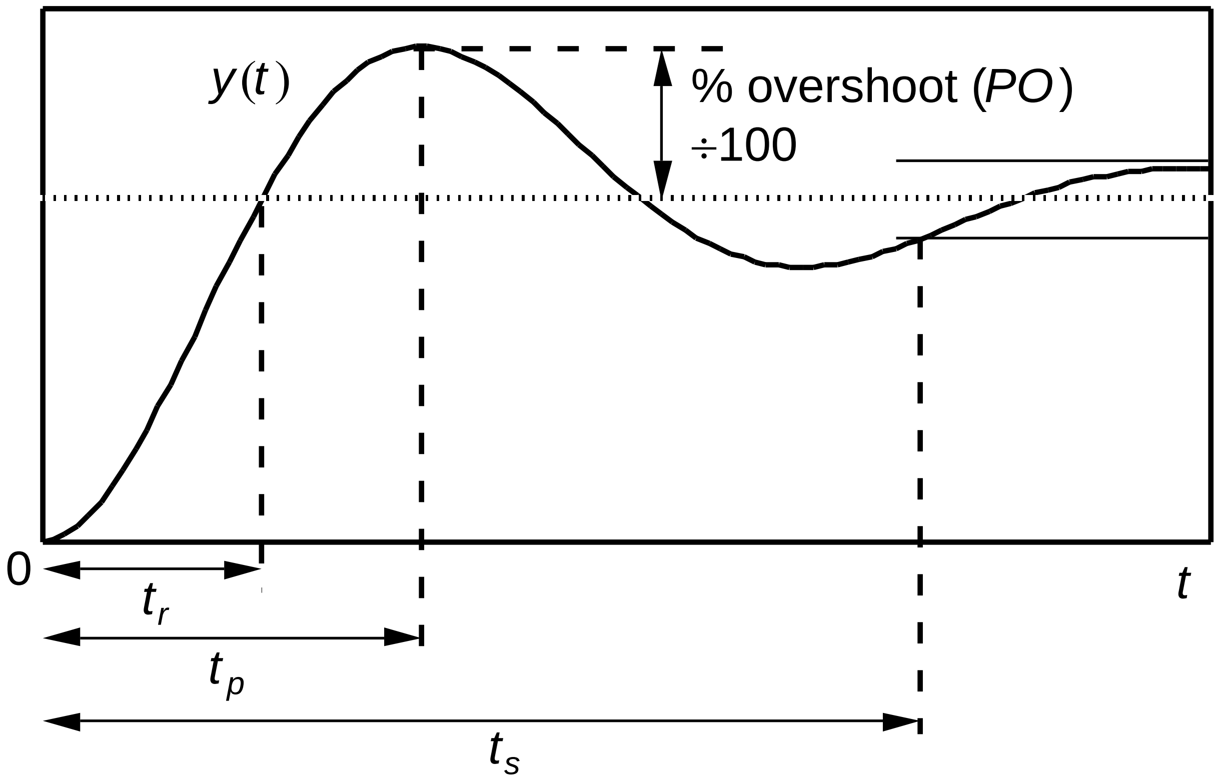
Sufficient conditions:

w^2:	2	1
w^1:	-1	0
w^0:	$\frac{-1}{-1}\begin{vmatrix} 2 & 1 \\ -1 & 0 \end{vmatrix} = 1$	

2 sign changes in the first column - 2 unstable poles

Time Domain Analysis

As with the continuous-time case, we can characterize the time response of digital systems by overshoot, rise time etc.



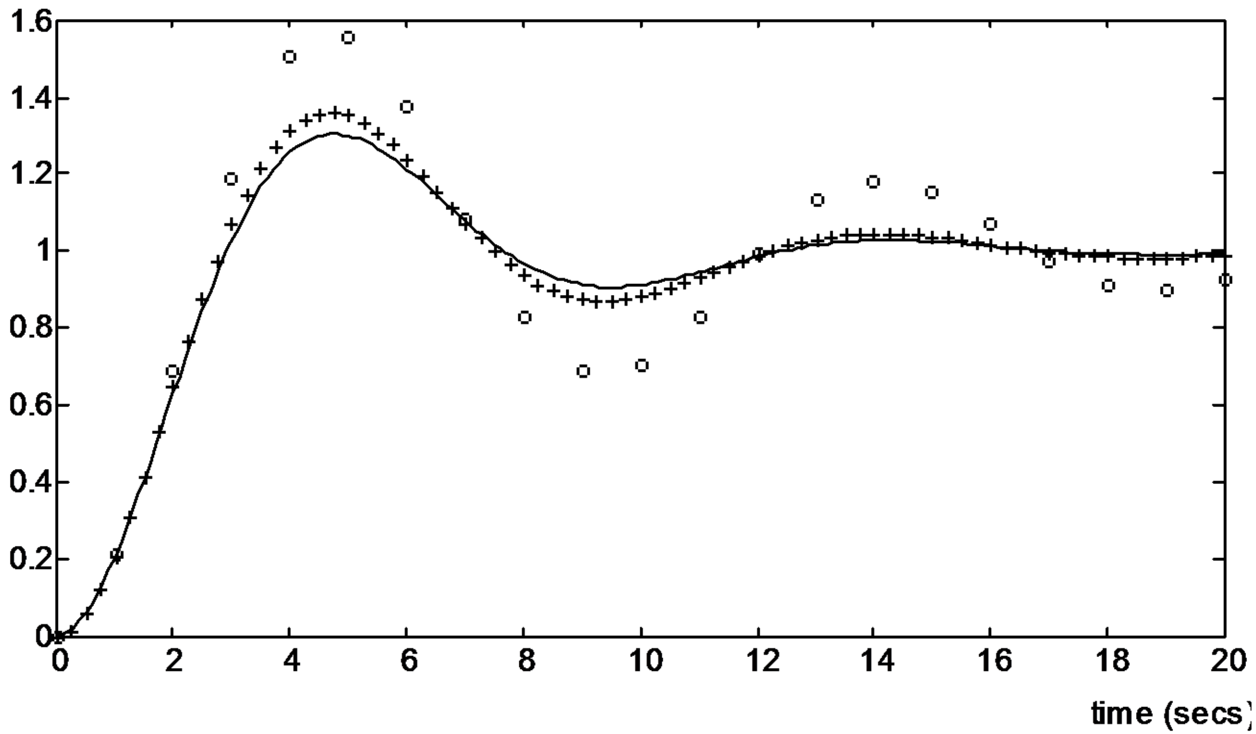
max. overshoot $\Rightarrow \zeta$ from standard 2nd order curve discrete-time case: care must be taken that sampling period is sufficiently small.

Example: - OLTF $G(s) = \dfrac{1}{s(1+2s)}$

Find CLTF and hence determine the closed loop unit step response.

For sampling times of 0.25 *s* and 1.0 *s*, find the OL *z* - transfer function, the CL *z* - transfer function and hence the CL unit step response.

Using MATLAB:



```
% MATLAB commands (code) to produce step responses, root loci,
% frequency responses for a continuous and discrete time system.
% (comments begin with a %)
clf reset;                          %clear all figures and reset properties
K=1;T=2;                            %system parameters
num=K;                              %open loop numerator
den=[T 1 0];                        %open loop denominator polynomial
cont_sys=zpk([],[0 -1/T],K/T);      %define continuous time system in
                                                               zero/pole/gain form

%calculate and display the closed loop system
cl_cont_sys=feedback(cont_sys,1)
pause

tfinal=20;                          %set final time
t=[0:0.1:tfinal];                   %produce time vector
figure(1)
step(cl_cont_sys,t)                 %plot now
hold on
pause
[y,x]=step(cl_cont_sys,t);          %store results for later

Ts1=0.25;                           %define sampling period
dis_sys1=c2d(cont_sys,Ts1,'zoh')    %discretize system
cl_dis_sys1=feedback(dis_sys1,1)    %close the feedback loop
step(cl_dis_sys1,tfinal)
pause
k1=[0:tf/Ts1];                         %"time" vector
[yz1,xz1]=step(cl_dis_sys1,tfinal);    %store results for later

Ts2=1.0;                               %increase sampling period
dis_sys2=c2d(cont_sys,Ts2,'zoh');      %repeat
cl_dis_sys2=feedback(dis_sys2,1)       %close the feedback loop
step(cl_dis_sys2,tfinal)
pause
hold off
k2=[0:tf/Ts2];                         %"time" vector
[yz2,xz2]=step(cl_dis_sys2,tfinal);    %store results for later

%Produce all plots together for comparison.
plot(t,y,k1*Ts1,yz1,'+',k2*Ts2,yz2,'o')
pause
```

```
%Now do root loci
figure(2)
rlocus(cont_sys)
pause
rlocus(dis_sys1)
axis('equal')
pause
rlocus(dis_sys2)
axis('equal')
pause

%Finally look at frequency response
figure(3)
bode(cont_sys)
hold on
pause
bode(dis_sys1)
pause
bode(dis_sys2)
pause
hold off

%Nyquist plots don't work very well due to type 1 system
% - need to restrict frequency range and specify axes.
figure(4)
wmin=0.1;                         %set minimum frequency
wmax=10;                          %set maximum frequency
nyquist(cont_sys,{wmin,wmax})
axis([-2 1 -2 1])                 %set axis limits
grid
hold on
pause
nyquist(dis_sys1,{wmin,wmax})
axis([-2 1 -2 1])
grid
pause
nyquist(dis_sys2,{wmin,wmax})
axis([-2 1 -2 1])
grid
pause
hold off
axis('normal')
```

```
close all
```

Root Locus Analysis

Block diagram algebra of closed - loop sampled - data systems leads to characteristic equations of the form,

$$1+G(z)H(z),\ 1+GH(z),\ \text{etc}$$

or in general $(1+P(z))$ where $P(z)$ is a formulation of the open - loop transfer function, the exact nature of which is determined by the position of samplers in the loop.

$P(z)$ is a rational function in z and therefore the characteristic polynomial can be written in standard pole - zero form as:

$$1+\frac{K\prod(z+z_i)}{\prod(z+p_i)} \qquad (1)$$

where z_i are the open loop system zeros, p_i the open loop system poles and K is a variable gain term.

Eq. (1) is in exactly the same form as can be obtained for root - locus analysis of characteristic polynomials in the s - domain and therefore the analysis is identical.

The only difference lies in the definition of the stability boundary.

Review of Root Locus

Although computer packages for plotting the root - locus are now readily available, it is important to know the basic rules for sketching the loci.

Re-writing (1) as

$$\frac{K(z-z_1)(z-z_2)\ldots(z-z_m)}{(z-p_1)(z-p_2)\ldots(z-z_n)} = -1 \qquad (2)$$

then any point on the root locus must satisfy the *magnitude condition*:

$$\frac{K|z-z_1||z-z_2|\ldots|z-z_m|}{|z-p_1||z-p_2|\ldots|z-p_n|} = 1 \qquad (3)$$

and the *angle condition*:

$$\begin{aligned}&\{\angle(z-z_1)+\angle(z-z_2)+\ldots\angle(z-z_m)\}\\ &-\{\angle(z-p_1)+\angle(z-p_2)+\ldots\angle(z-p_n)\} = i\pi \qquad (4)\\ &i = \ldots-3,-1,1,3,5,\ldots\end{aligned}$$

The angle condition is used to locate points on the root locus and the magnitude condition gives the value of *K* at that point.

Manual sketching of the root - locus diagram is considerably eased by a series of rules that when methodically applied give a good indication of the shape of the loci.

1) The loci start (i.e. $K = 0$) at the n poles of the open loop TF $P(z)$

2) The no. of loci is equal to the order of the characteristic equation. (The plot is symmetrical about the real axis.)

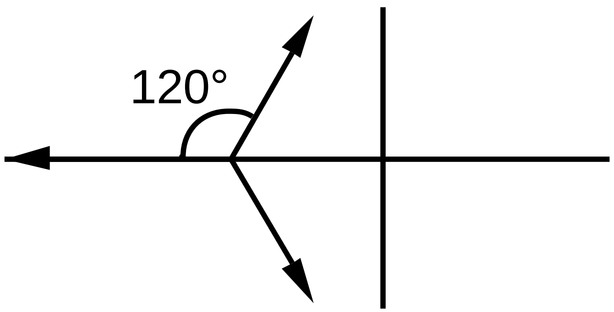
3) The root loci end (i.e. $K \to \infty$) at the m zeros of $P(z)$, and if $m < n$ (usual) then the remaining n - m loci tend to infinity.

4) Portions of the real axis are sections of a root locus if the no. of poles and zeros lying on the axis to the right is odd.

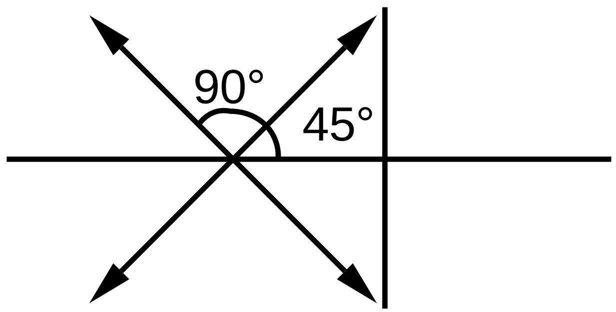
5) Those loci terminating at infinity tend towards asymptotes at angles relative to the positive real axis given by:

$$\frac{\pi}{(n-m)}, \frac{3\pi}{(n-m)}, \frac{5\pi}{(n-m)}, \dots, \frac{\{2(n-m)-1\}\pi}{(n-m)} \quad (5)$$

Examples:



3 excess poles



4 excess poles

6) The intersection of the asymptotes on the real axis occurs at the 'centre of gravity' of the pole - zero configuration of $P(z)$, i.e. at

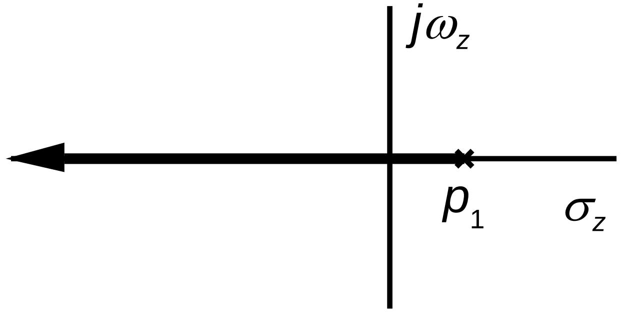
$$z = \frac{\sum \text{poles of } P(z) - \sum \text{zeros of } P(z)}{(n-m)} \qquad (6)$$

7) The intersection of the root - loci with the unit circle can be calculated by Jury, Bilinear Transformation/Routh - Hurwitz or geometrical analysis (only on some plots).

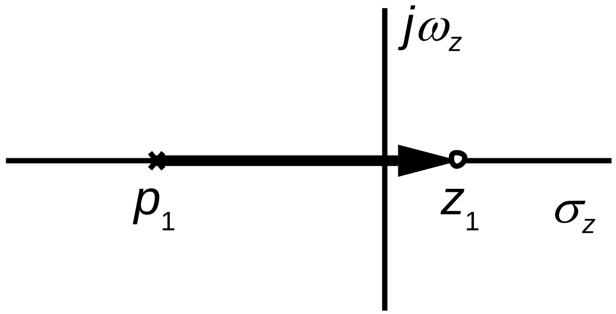
8) The breakaway points (points at which multiple roots of the characteristic polynomial occur) of the root locus are the solutions of $\frac{dK}{dz} = 0$ (not all the solutions are necessarily breakaway points.)

Examples

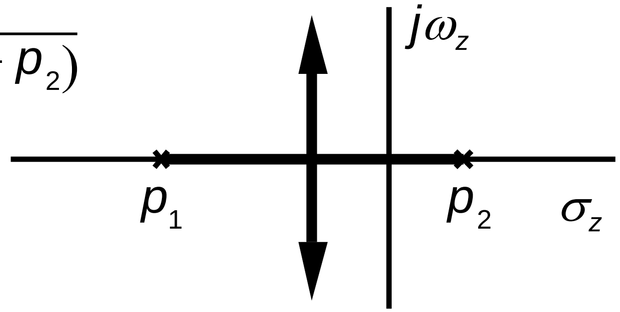
$$P(z) = \frac{1}{z - p_1}$$



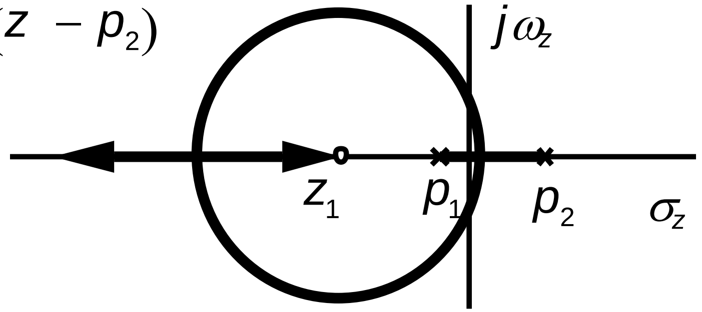
$$P(z) = \frac{z - z_1}{z - p_1}$$



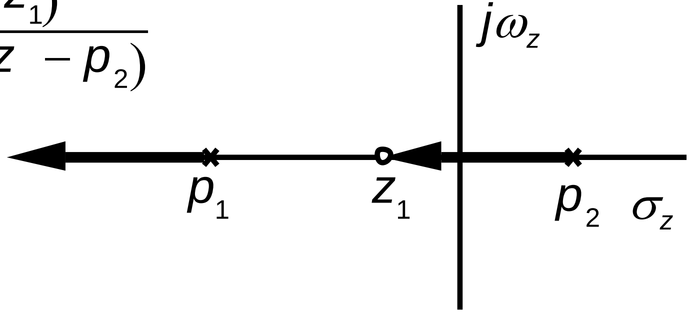
$$P(z) = \frac{1}{(z - p_1)(z - p_2)}$$



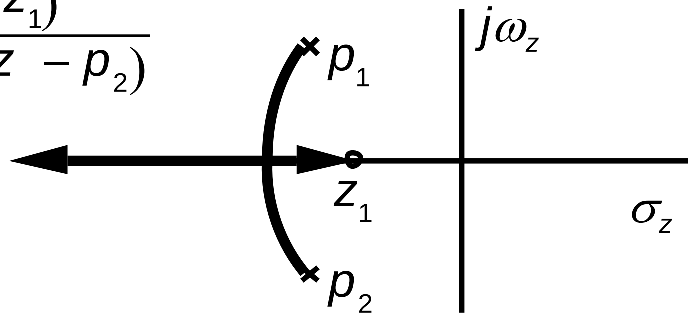
$$P(z) = \frac{(z - z_1)}{(z - p_1)(z - p_2)}$$



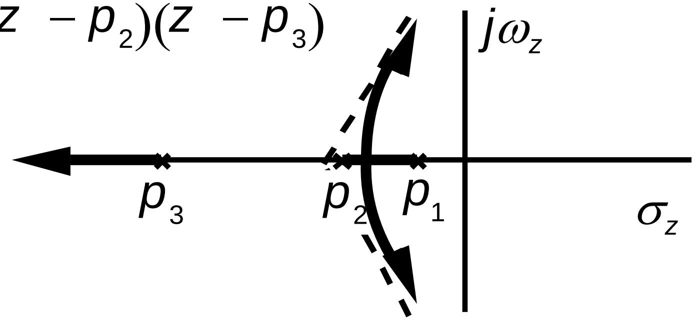
$$P(z) = \frac{(z - z_1)}{(z - p_1)(z - p_2)}$$



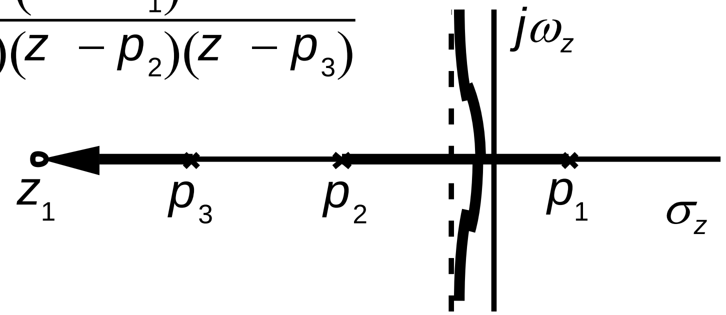
$$P(z) = \frac{(z - z_1)}{(z - p_1)(z - p_2)}$$



$$P(z) = \frac{1}{(z - p_1)(z - p_2)(z - p_3)}$$

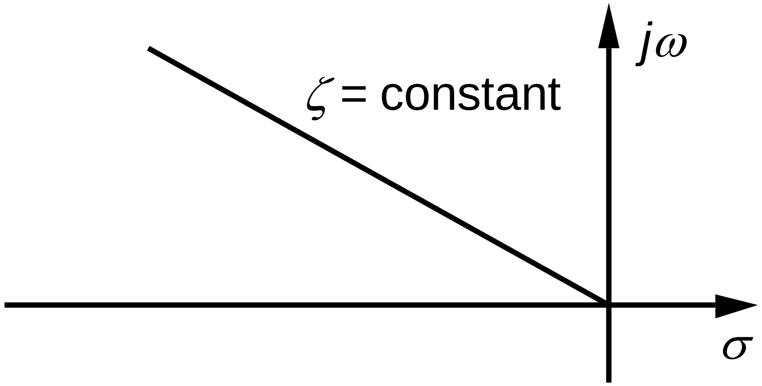


$$P(z) = \frac{(z - z_1)}{(z - p_1)(z - p_2)(z - p_3)}$$



Lines of constant damping ratio, ζ:

in the s-plane, constant ζ is represented by:



$$s = -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}$$

using the mapping $z = e^{sT}$

$$z = e^{-\zeta\omega_n T + j\omega_n T\sqrt{1-\zeta^2}} = e^{-\zeta\omega_n T} e^{j\omega_n T\sqrt{1-\zeta^2}}$$

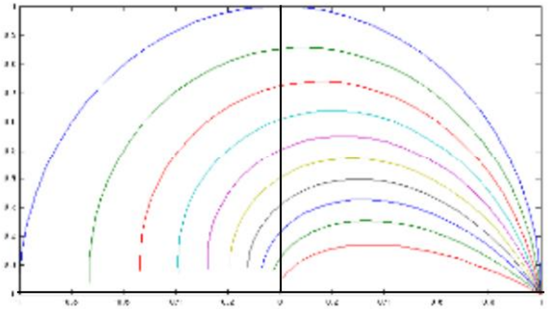
so $\quad |z| = e^{-\zeta\omega_n T} \quad$ and $\quad \angle z = \omega_n T\sqrt{1-\zeta^2}$

For fixed ζ, as ω_n increases:

$|z|$ decreases exponentially; $\angle z$ increases linearly

i.e. logarithmic spiral

Lecture 7b – Continuation of Lecture 7a



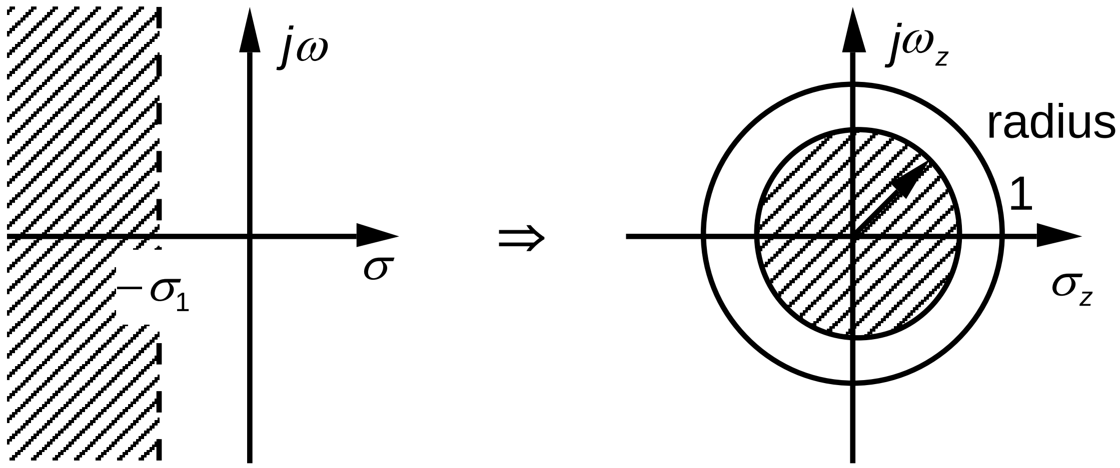
Settling time considerations:

For settling to within 5%, $t_s = \dfrac{3}{\zeta\omega_n}$ - depends on $\zeta\omega_n$

So real part of $\{s\} \leq -3/t_s$

In the z-plane, $|z| = e^{-\zeta\omega_n T}$

$\therefore \quad |z| \leq e^{-3T/t_s}$



Digital Control System Design

General requirements:

- stability of the closed-loop system
- good transient behaviour
- good steady state behaviour
- good disturbance rejection

+

the control algorithm must be realizable

i.e. not require future values of control signals

Design Methods

- continuous-time design followed by digital re-design
- digital frequency domain design
- digital root locus design
- state feedback design
- digital PID design

- deadbeat response design

Digital Re-Design of Continuous-Time Controllers

- one of the simplest methods

Procedure:

design continuous compensator using traditional methods
(Bode, phase lead etc)
then “discretize” the resulting compensator.

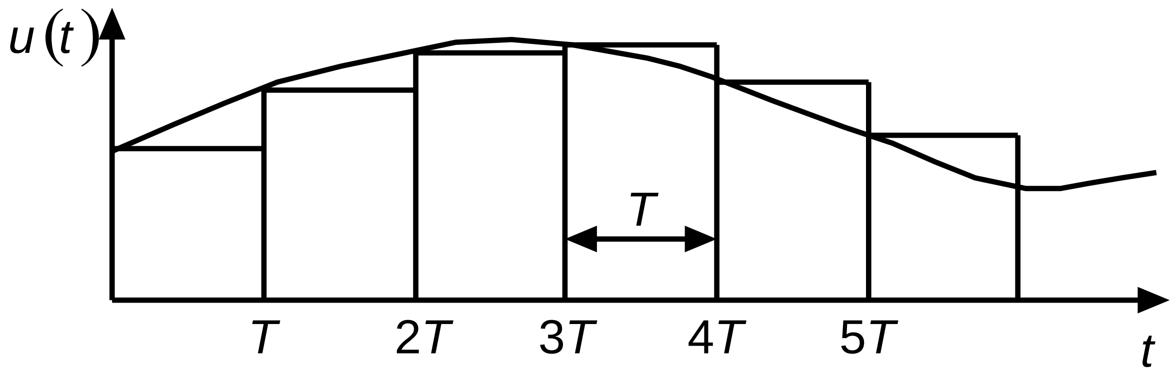
Traditionally popular in industry

- continuous methods are well understood

- many processes have existing continuous-time compensators

1 a) Numerical Integration: Forward Rectangular Rule

consider a continuous variable $u(t)$:



if $y(t)=\int u(t)dt =$ area under curve, then $D(s)=\dfrac{Y(s)}{U(s)}=\dfrac{1}{s}$

$\int u(t)dt \approx$ area of rectangles

i.e. $y(kT)=y[(k-1)T]+Tu[(k-1)T]$

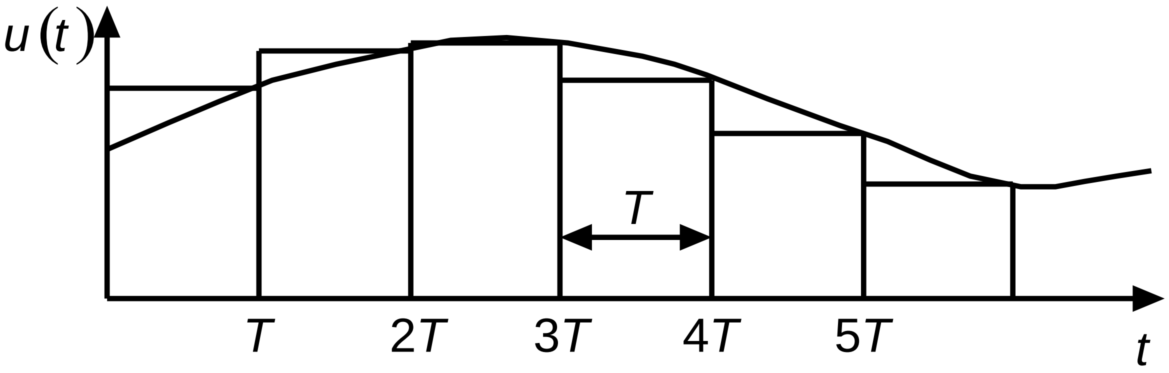
e.g. $y(4T)=y(3T)+$ new area

so $D(z)=\dfrac{Y(z)}{U(z)}=\dfrac{Tz^{-1}}{1-z^{-1}}=\dfrac{T}{z-1}$

$\therefore D(z)$ is obtained from $D(s)$ by making the substitution:

$$s = \frac{z - 1}{T}$$

1 b) Numerical Integration: Backward Rectangular Rule



so $y(kT) = y[(k-1)T] + Tu(kT)$

$$\therefore D(z) = \frac{Y(z)}{U(z)} = \frac{T}{1-z^{-1}} = \frac{Tz}{z-1}$$

so $D(z)$ is obtained from $D(s)$ by making the substitution:

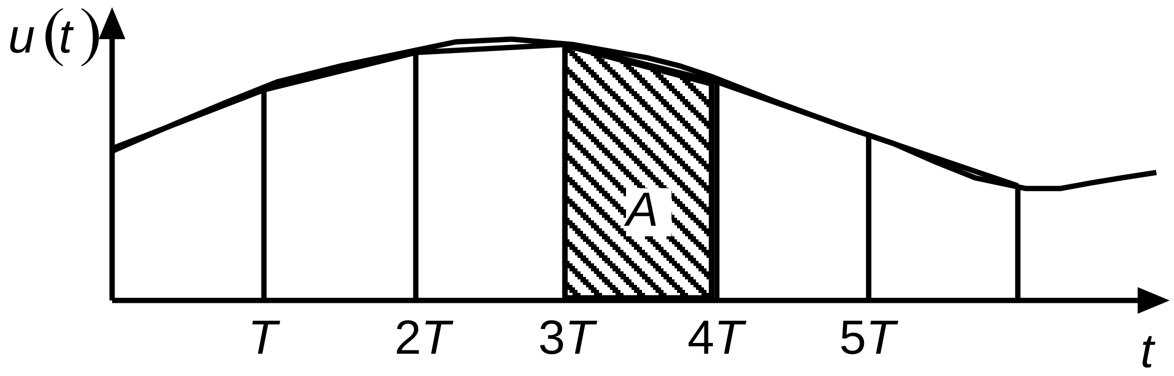
$$s = \frac{z-1}{Tz}$$

alternatively, can also consider

$$sY(s) \quad \Rightarrow \quad \frac{dy}{dt} \approx \frac{\Delta y}{\Delta t} = \frac{y(kT) - y[(k-1)T]}{T} \quad \Rightarrow \quad \frac{1-z^{-1}}{T} Y(z)$$

finite difference

1 c) Numerical Integration: Tustin's Rule



$y(t) = \int u(t)dt =$ area of trapeziums

so $y(kT) = y[(k-1)T] + \frac{T}{2}\left(u(kT) + u[(k-1)T]\right)$

i.e. $y(4T) = y(3T) + A$

$\therefore\ D(z) = \frac{Y(z)}{U(z)} = \frac{T}{2}\frac{1+z^{-1}}{1-z^{-1}} = \frac{T}{2}\frac{z+1}{z-1}$

remembering $D(s) = \frac{1}{s}$, $D(z)$ is obtained from $D(s)$ by making the substitution:

$s = \frac{2}{T}\frac{z-1}{z+1}$ Tustin's rule

(bilinear transformation)

Stability Considerations of Numerical Integration Rules

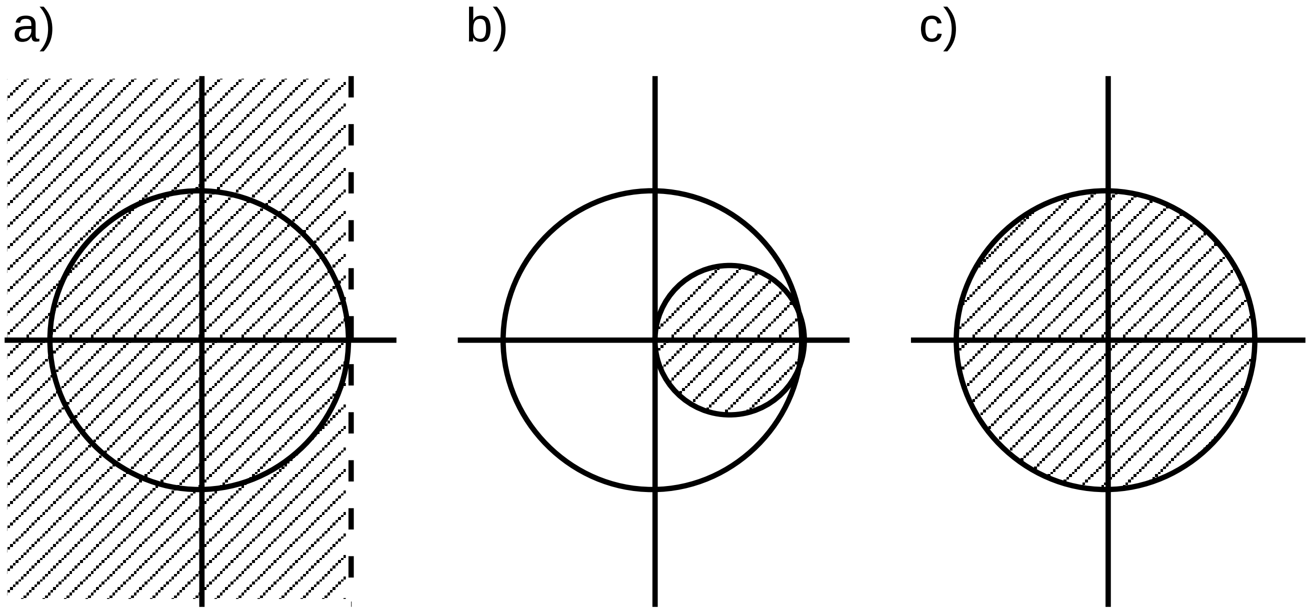
The following relations can be derived:

a) forward rectangular rule: $z = 1 + Ts$

b) backward rectangular rule: $z = \dfrac{1}{1 - Ts}$

c) Tustin's rule: $z = \dfrac{(1 + Ts/2)}{(1 - Ts/2)}$

Letting $s = j\omega$ gives the stability boundary for each approximation:



a stable $D(s)$ could give

unstable $D(z)$

Pre-warping with Tustin's Rule

The stability boundary using Tustin's rule is the same as $z = e^{sT}$

BUT the complete $j\omega$ axis is mapped into the 2π circumference of the unit circle which is not the case for the mapping $z = e^{sT}$

$\therefore$ a large amount of frequency distortion occurs

A measure of the distortion can be obtained by considering the relationship between ω_c in the *s*-domain and ω_d in the *z*-domain

i.e. $s = j\omega_c$ and $z = e^{j\omega_d T}$

From Tustin's rule: $j\omega_c = \dfrac{2}{T}\dfrac{\left(e^{j\omega_d T} - 1\right)}{\left(e^{j\omega_d T} + 1\right)}$

$$\therefore \quad j\omega_c = \frac{2}{T}\frac{e^{j\omega_d T/2} - e^{-j\omega_d T/2}}{e^{j\omega_d T/2} + e^{-j\omega_d T/2}} = \frac{2}{T}\frac{2j\sin(\omega_d T/2)}{2\cos(\omega_d T/2)}$$

$$\therefore \quad \omega_c = \frac{2}{T}\tan(\omega_d T/2)$$

The distortion can be eliminated for a particular frequency of interest, ω_a by modifying Tustin's rule:

$$s = \frac{\omega_a}{\tan(\omega_a T/2)} \frac{(z-1)}{(z+1)}$$

2. Pole-Zero Mapping

An alternative approach is to use the mapping $z = e^{sT}$.

For a continuous signal $y(t)$, the poles of the Laplace transform $Y(s)$ are related to the poles of the z-transform $Y(z)$ of $y(kT)$ by $z = e^{sT}$.

This is NOT true for the zeros of $Y(s)$ and $Y(z)$ and the z-transform must be obtained to locate the zeros.

For small T, $z = e^{sT}$ is approximately true for the zeros as well as the poles.

Heuristic Rules:

1. All poles of $D(s)$ are mapped according to $z = e^{sT}$

   $\left[s + p = 0 \quad \Rightarrow \quad z - e^{-pT} = 0\right]$

2. All finite zeros of $D(s)$ are also mapped according to $z = e^{sT}$

   $\left[s + \zeta = 0 \quad \Rightarrow \quad z - e^{-\zeta T} = 0\right]$

3. EITHER: the q zeros of $D(s)$ at $s = \infty$, where q is the pole excess, are mapped to $z = -1$ (i.e. $D(z)$ zeros at $z = -1$)

   OR: $q - 1$ zeros of $D(s)$ at $s = \infty$ are mapped to $z = -1$ and the remaining zero at $s = \infty$ is mapped to a zero at $z = \infty$

   (This leads to a $D(z)$ which has a unit delay in its impulse response.)

4. The gain of $D(z)$ is selected to match the gain of $D(s)$ at the band centre, or a similar critical frequency. In most control applications, the critical frequency is $s = 0$ so typically:

$$D(z)\big|_{z=1} = D(s)\big|_{s=0}$$

3. Zero-order Hold Equivalence

A final approach is to consider a transformation such that $D(s)$ and $D(z)$ have identical step responses at the sampling instants

i.e. $$\mathcal{Z}^{-1}\left\{D(z)\frac{1}{1-z^{-1}}\right\} = \left[\mathcal{L}^{-1}\left\{D(s)\frac{1}{s}\right\}\right]_{t=kT}$$

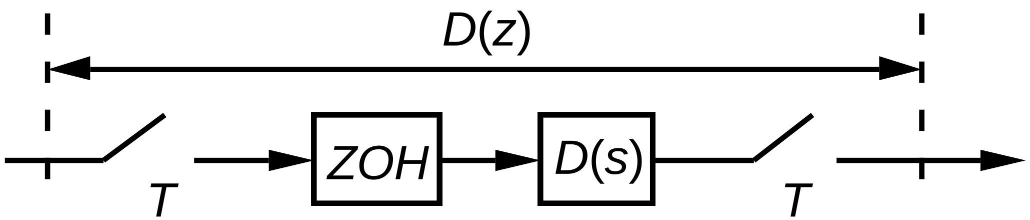
or $$D(z)\frac{1}{1-z^{-1}} = \mathcal{Z}\left\{\mathcal{L}^{-1}\left\{D(s)\frac{1}{s}\right\}\right\}_{t=kT}$$

$$= \mathcal{Z}\left\{D(s)\frac{1}{s}\right\}$$

$$\therefore \quad D(z) = \left(1-z^{-1}\right)\mathcal{Z}\left\{D(s)\frac{1}{s}\right\}$$

The MATLAB command c2d converts from continuous to discrete-time:
- the default method is ‘zoh’
- other methods are:
  - ‘foh’
  - ‘tustin’
  - ‘prewarp’
  - ‘matched’

i.e.



This is probably the most popular approach.

Example: Digital Re-design of $D(s) = \dfrac{s}{s^2 + s + 25}$

{bandpass filter with a centre frequency of $\omega_0 = 5\,rads^{-1}$}

First determine a suitable sampling period T:

- consider the frequency response of the continuous system at 1.5 Hz;

$|D(j\omega)| < 0.2$ (i.e. < -14 dB)

$\therefore$ suitable sampling frequency $f_s = 2 \times 1.5 = 3\,Hz$ (but could be higher)

$$\therefore \quad T = \frac{1}{3}\,sec$$

---

1. a) Forward rectangular rule (FR)

$$\frac{s}{s^2 + s + 25} \Rightarrow \frac{(z-1)/T}{(z^2 - 2z + 1)/T^2 + (z-1)/T + 25}$$

$$= \frac{z-1}{3z^2 - 5z + 10.33} = \frac{0.3333z - 0.3333}{z^2 - 1.6667z + 3.4444} \quad \text{(unstable)}$$

1. b) Backward rectangular rule (BR)

$$\frac{s}{s^2+s+25} \Rightarrow \frac{(z-1)/Tz}{(z^2-2z+1)/T^2z^2+(z-1)/Tz+25}$$

$$=\frac{z(z-1)}{12.33z^2-7z+3} = \frac{0.0811z^2-0.0811z}{z^2-0.5676z+0.2432}$$

1. c) Tustin's rule (TU)

$$\frac{s}{s^2+s+25} \Rightarrow \frac{\frac{2}{T}\frac{(z-1)}{(z+1)}}{\frac{4}{T^2}\frac{(z-1)^2}{(z+1)^2}+\frac{2}{T}\frac{(z-1)}{(z+1)}+25}$$

$$=\frac{0.6667z^2-0.6667}{7.444z^2-2.444z+6.111} = \frac{0.0896z^2-0.0896}{z^2-0.3283z+0.8209}$$

1. d) Tustin's rule with Prewarping (TUW)

$$\frac{s}{s^2+s+25} \Rightarrow \frac{\frac{\omega_a}{\tan(\omega_a T/2)}\frac{(z-1)}{(z+1)}}{\frac{\omega_a^2}{\tan^2(\omega_a T/2)}\frac{(z-1)^2}{(z+1)^2}+\frac{\omega_a}{\tan(\omega_a T/2)}\frac{(z-1)}{(z+1)}+25}$$

$$= \frac{5.5029z^2 - 5.5029}{60.8z^2 + 10.6 + 49.8} = \frac{0.0905z^2 - 0.0905}{z^2 + 0.1741z + 0.8189} \qquad (\omega_a = \omega_0)$$

2. Pole-Zero Mapping (PZ)

$$\frac{s}{s^2+s+25} = \frac{s}{(s+0.5-j4.97)(s+0.5+j4.97)}$$

$$\Rightarrow \frac{K_{\omega_a}(z-1)}{\left(z-e^{(-0.5+j4.97)T}\right)\left(z-e^{(-0.5-j4.97)T}\right)}$$

$$= \frac{0.1909z-0.1909}{z^2+0.148z+0.7165}$$

3. Zero Order Hold Equivalence (ZOH)

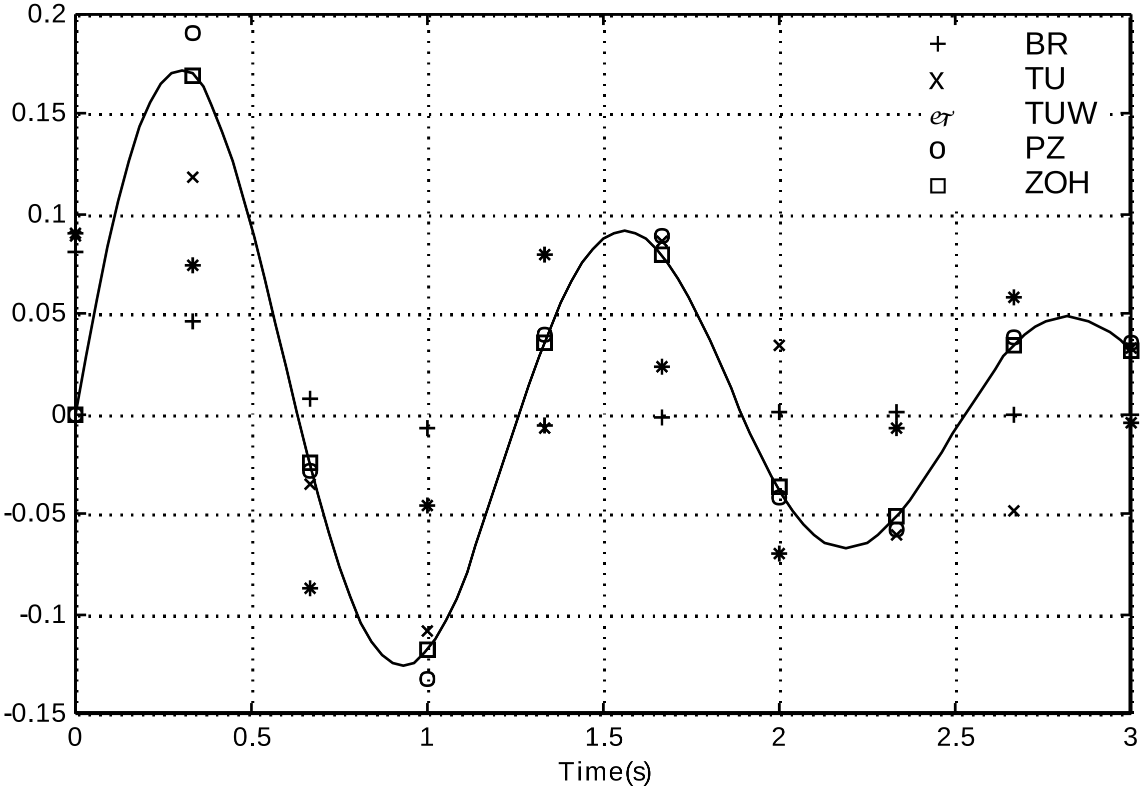
$$\frac{s}{s^2+s+25} \Rightarrow \left(1-z^{-1}\right)\mathcal{Z}\left\{\frac{1}{s^2+s+25}\right\}$$

$$=\left(1-z^{-1}\right)\mathcal{Z}\left\{\frac{-j\frac{1}{2\times 4.97}}{(s+0.5-j4.97)}+\frac{j\frac{1}{2\times 4.97}}{(s+0.5+j4.97)}\right\}$$

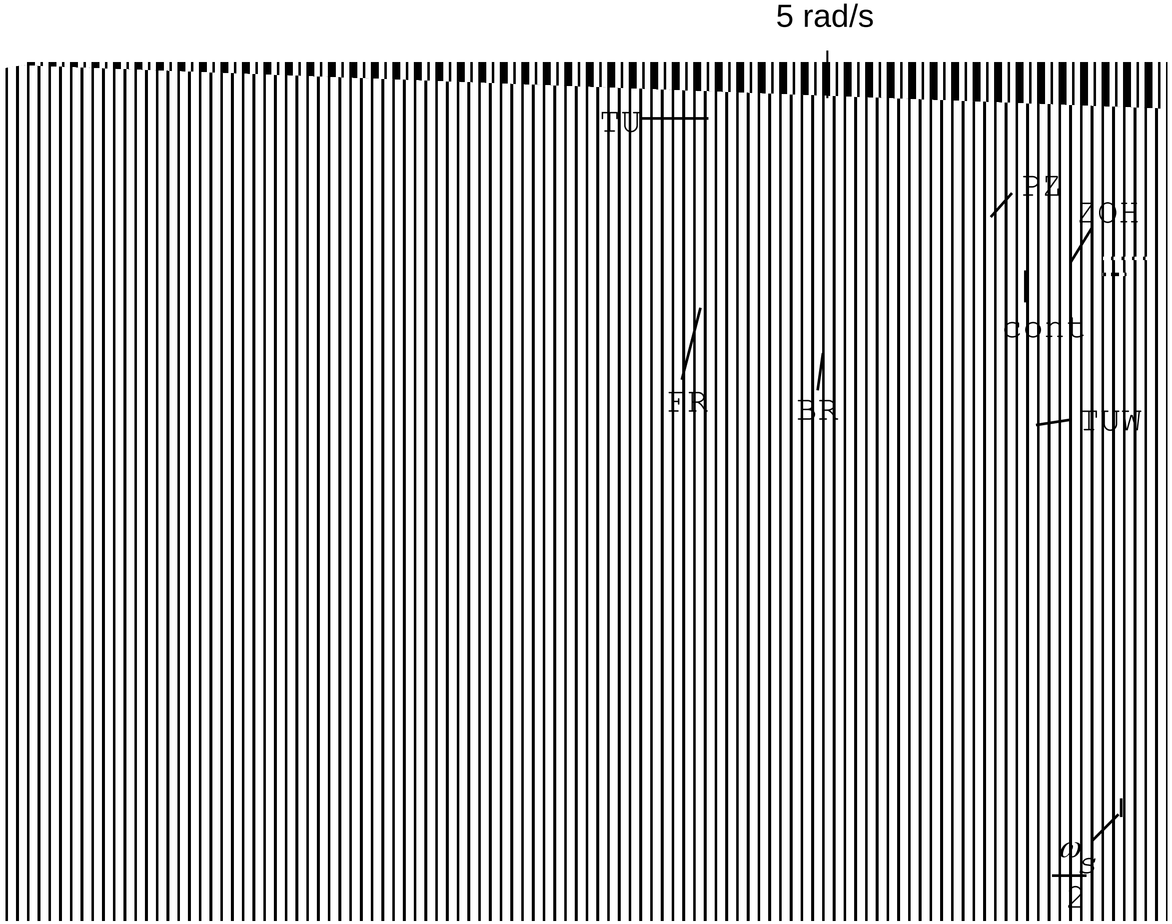
$$=\left(1-z^{-1}\right)j\,\frac{1}{2\times 4.97}\left[\frac{z}{\left(z-e^{(-0.5-j4.97)T}\right)}-\frac{z}{\left(z-e^{(-0.5+j4.97)T}\right)}\right]$$

$$= \frac{0.1695z-0.1695}{z^2+0.148z+0.7165}$$

Step Responses for Various Mappings



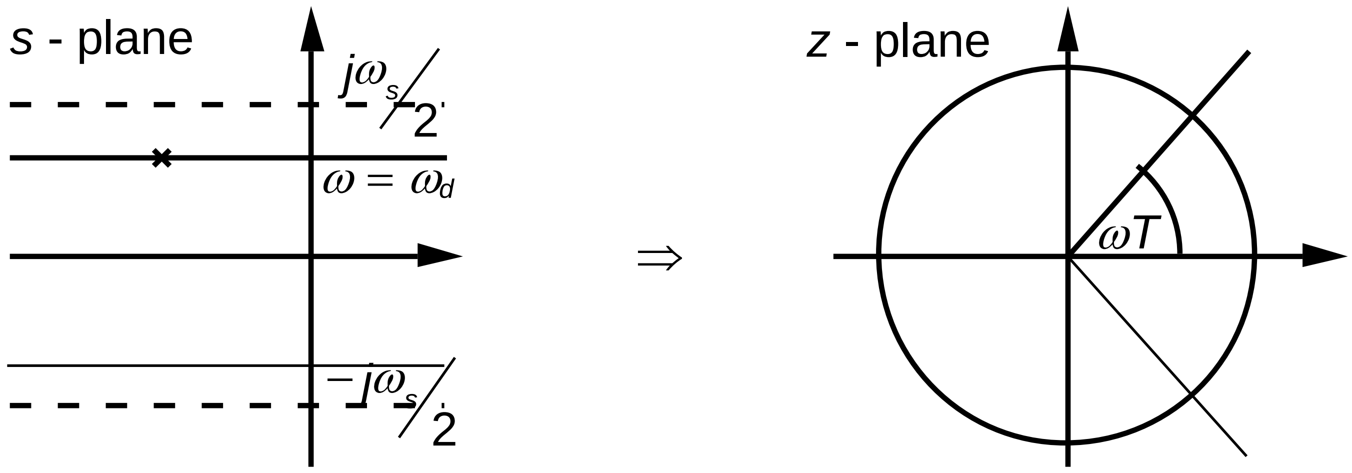
Bode Magnitude Plots for Various Mappings



Digital Root Locus Design

Again we've seen analysis using root locus in the *z* - plane
- design is very similar to *s* - plane.

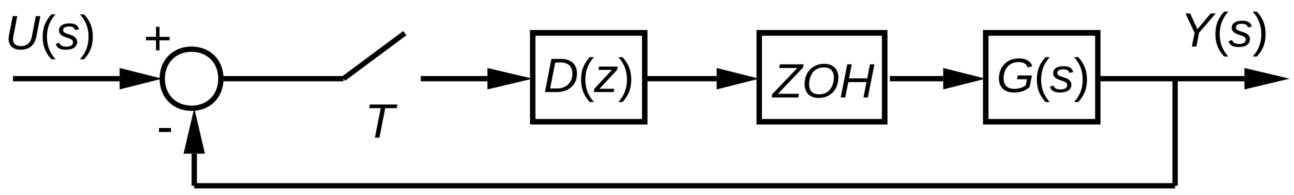
There is one further consideration to do with the sampling rate:



Hence the angle of a particular pole location gives the number of samples/cycle in the time response.

Example: Design a compensator for a system with TF $G(s) = \frac{1}{s^2}$

such that $\zeta \geq 0.5$ and $t_s(\pm 2\%) \leq 1$ sec for a step input.



From the specification, $f_d = f_n\sqrt{1-\zeta^2} \approx 1 Hz$

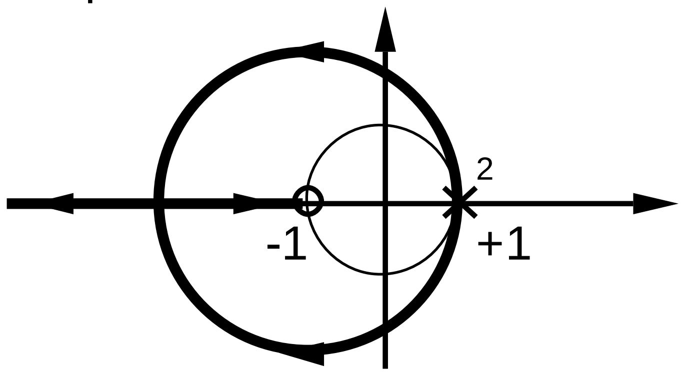
$\therefore$ choose sampling frequency 10 - 20 times higher, say $f_s = 20\,Hz$

$\therefore$ $T = 0.05\,\text{sec}$

- open loop z-transfer function:

$$G(z) = \mathcal{Z}\left\{\frac{1-e^{-sT}}{s^3}\right\} = \frac{T^2}{2}\frac{(z+1)}{(z-1)^2} = 0.00125\frac{(z+1)}{(z-1)^2}$$

The root locus in the z-plane is:



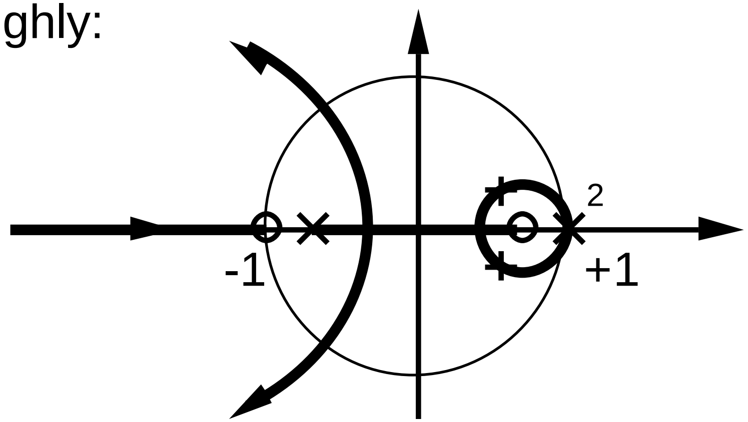
We require a zero near the double pole to "pull" the root locus inside the unit circle.

Settling time: poles must lie within a circle of radius $e^{-4T/t_s} \approx 0.8$

- choose CL poles such that $|z| \approx 0.7$ and $\zeta = 0.6$ locus

Try $D(z) = K\dfrac{(z-0.7)}{(z+0.8)}$ pole required for "proper" controller (but well away from area of interest)

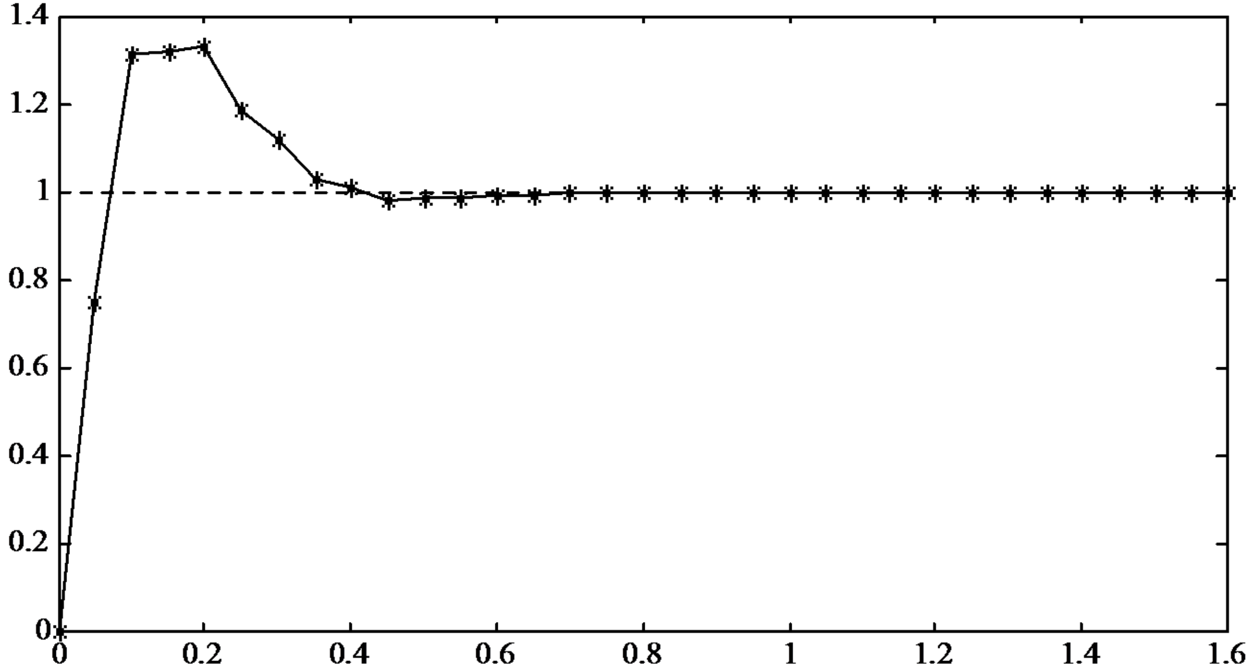
Roughly:



requires K = 600

Lecture 7b – Continuation of Lecture 7a

Results:



overshoot is too large but t_s is well within specification

- repeat design using slower settling time

i.e. $D(z) = 240\dfrac{(z-0.9)}{(z+0.8)}$

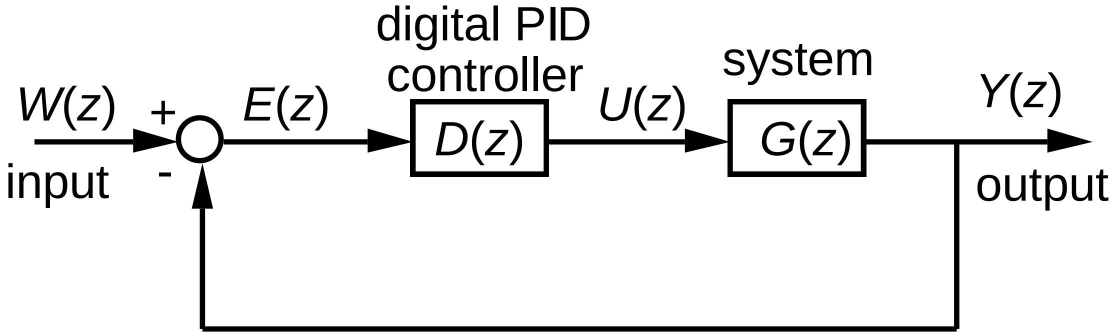
Digital PID Control Design

* process industries commonly use "3-term" (PID) controllers

$$D(s) = K_P + \frac{K_I}{s} + K_D s$$

or

$$D(s) = K\left(1 + \frac{1}{T_I s} + T_D s\right)$$



Consider the control system:

The control signal (output of the 3-term controller) is:

$$u^*(kT) = K_P e^*(kT) + K_I i^*(kT) + K_D d^*(kT)$$

where *i** and *d** are the integral and derivative of the error respectively

We need to approximate $i^*(kT)$ and $d^*(kT)$.

Two Methods

- Euler's method:

$$i^*(kT) = i^*((k-1)T) + T\,e^*(kT) \qquad \text{[backward rectangular rule]}$$

- Trapezium rule:

$$i^*(kT) = i^*((k-1)T) + \frac{T}{2}\left(e^*(kT) + e^*((k-1)T)\right) \qquad \text{[Tustin's rule]}$$

Also: $$d^*(kT) = \frac{e^*(kT) - e^*((k-1)T)}{T}$$

In the z - domain:

(i) $$I(z) = \frac{Tz}{z-1}E(z)$$

(ii) $$I(z) = \frac{T}{2}\frac{(z+1)}{(z-1)}E(z)$$

and $$D(z) = \frac{z-1}{zT}E(z)$$

Now we need to find values for K_P, K_I and K_D

Ziegler - Nichols Method: closed loop

- empirically based, derived from studies of “perfect” systems.

Method:

Use proportional CL control only, start with a low gain and increase until plant output oscillates with constant amplitude.

- call the period of oscillation T_u and value of gain K_u.

The gains to give “good” responses are:

- Proportional (P) Control only:

$$K_P = 0.5K_u$$

- Proportional + Integral (PI) Control:

$$K_P = 0.45K_u \qquad K_I = \frac{1.2K_P}{T_u} = \frac{0.54K_u}{T_u}$$

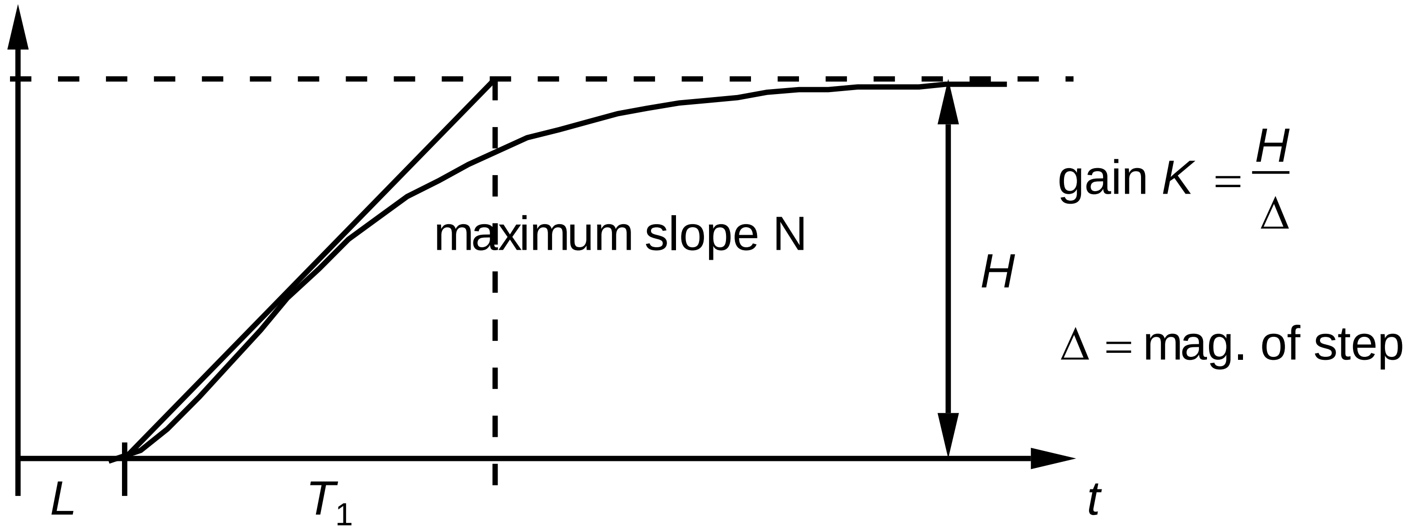
- Proportional + Integral + Derivative (PID) Control:

$$K_P = 0.6K_u \qquad K_I = \frac{1.2K_u}{T_u} \qquad K_D = \frac{0.6T_uK_u}{8}$$

Only applicable for systems which are CL stable at low gains.

Ziegler - Nichols Method: open loop

- basically the same as the continuous time design.



Open loop step response:

- P control: $K_P = \frac{\Delta}{NL} = \frac{\Delta T_1}{HL} = \frac{T_1}{KL}$
- PI control: $K_P = \frac{0.9\Delta}{NL}$ $K_I = \frac{0.3K_P}{L}$
- PID control: $K_P = \frac{1.2\Delta}{NL}$ $K_I = \frac{0.5K_P}{L}$ $K_D = 0.5LK_P$

Only works for stable OL type 0 systems