Sequences, Series and Taylor Approximation (MA2712b, MA2730)

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November 20, 2015

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Chapter 0

Introduction, Overview

The first 12 lectures (Chapters 1-3) contribute to the study blocks MA2730 (for M, FM, MSM and MCS) and MA2712b (for MMS and MCC). Those blocks feed into the assessment blocks MA2812 and MA2815 (for M, FM, MSM and MCS) or MA2810 (for MMS and MCC). The purpose of those lectures is to make you familiar with important concepts in Calculus and Analysis, namely those of **sequences** and **series** as well as **Taylor**[1] **polynomials** and **series**. In the first three chapters, you shall be introduced to elementary ideas about these concepts, so you could apprehend them as well as follow and perform relevant calculations. Students studying the full Analysis study block (MA2730) will continue, revisiting, broadening and deepening these concepts. In particular, you will be given the means to use more formal definitions and prove the results stated in the following first set of lectures.

Most of Calculus (MA1711) and Fundamentals of Mathematics (MA1712) may be used in this set of lectures. For quick reference, we have put some essential background material of Level 1 in a revision section on Blackboard. You will have two lectures a week with one seminar (the class is split in four seminar groups). The 24 lectures are split into 6 chapters; each section of those chapters corresponding to a lecture. There will be one set of exercise sheets per chapter, obviously most of these sheets lasting for a few weeks. In Exercise Sheet number Na, $\mathrm{N} = 1, \ldots, 6$, we expect to give you some short feedback first (to check your answers), finally rather detailed feedback. Additional exercises are given in Exercise Sheets Nb, $\mathrm{N} = 1, \ldots, 6$. Hence there will only be short feedback for them.

Chapter 1 Taylor Polynomials (5 Lectures)

Taylor polynomials are **approximating** a given differentiable function f, say, in a neighbourhood of a given point $x = a$, say, by a polynomial of degree n, say. This means that the notation $T_n^a f$ for such polynomial looks cumbersome, but it encodes the full information we need to know about them. Because polynomials are often easier to manipulate than more complex functions, we can replace those by one of their appropriate Taylor polynomial

[1]B. Taylor (1685-1731) was an English mathematician who worked on analysis, geometry and mechanics (vibrating string). He introduced the 'calculus of finite differences'. In that context, he invented integration by parts and gave his version of what is known as Taylor's Theorem, although the theorem was already known. He got embroiled in the disputes between English and continental mathematicians about following Newton's or Leibnitz's versions of calculus. He worked also on the foundation of descriptive and projective geometry but did not elaborate although he gave some deep results.

1. to evaluate limits,

2. to determine the type of a degenerate critical point or, even,

3. to approximate integrals of complicated functions.

In the first two operations, the result of the replacement will be an **exact** value, in the last it will only be an **approximation**.

In the five lectures of this chapter, we shall first define Taylor polynomials, then give examples of calculation of those Taylor polynomial. This involves the calculation of many derivatives. You should pay attention to the **Product, Quotient** and **Chain Rules** for differentiation. In the second lecture we shall also give a first idea about the **error** $R_n^a f = f - T_n^a f$ made by replacing a function f by one of its Taylor polynomial $T_n^a f$. In practical term, the calculation of a Taylor polynomial of a complex function can be simplified by using calculus rules to obtain the **calculus of Taylor polynomials**. To justify those calculations we need the information we got in the second lecture about the error term. Those are the topics of the third lecture. We end the chapter with two lectures on the application of Taylor polynomials to **calculate limits** and to study the **type of degenerate critical points** of real functions.

Chapter 2: Real Sequences (3 Lectures)

In the second chapter, we consider **sequences** (of real numbers) and their **limits**. Sequences are **ordered countable sets** $a_1, a_2, \ldots$ of real numbers. In this chapter we shall concentrate on special sequences defined by real functions such that $a_n = f(n)$, for all $n \in \mathbb{N}$, where $f : [1, \infty) \to \mathbb{R}$. This will allow us to use what you did with limits of functions at Level 1. This is not a significant restriction because, in practice, we shall use sequences of this type. In lecture six, the limit of such sequence will then be defined as equal to $\lim_{x\to\infty} f(x)$. In lecture seven, we use the **algebra of limits of functions** to determine an essentially equivalent **algebra of limits of sequences** as well as establishing some **important limits**. Those results are important because, **when they hold**, they free us to have to go back to the initial definition. The chapter ends with lecture eight where we discuss some important **qualitative properties** of sequences, in particular the fundamental theorem stating that **bounded AND monotone sequences converge**.

Chapter 3: Improper Integrals (2 Lectures)

This material has been pulled through from Level 1 in order to leave more room behind. It deals with the evaluation of definite integrals in two cases. Type 1: where the interval of integration if infinite; and, Type 2: where the integrand becomes infinite at a vertical asymptote.

Chapter 4: Real Series (4 Lectures)

The first half of the course finishes in Chapter 3 with the study of **series** of real numbers. Series are **infinite sums of sequences**, say $\sum_{n=1}^{\infty} a_n$. You have already seen that a geometric series converges if and only if its common ratio is strictly smaller than 1 in modulus. In lecture nine, we give the definition of the **convergence of a series**, using the convergence of the **sequence of partial sums**. In the next lecture ten, we look at other important series and show that there exists an **algebra of convergent series**. This leads to the discussion of many tests to assess the convergence or divergence of series **without calculating their sum**. In the last two lectures of the chapter, eleven and twelve, we discuss the **Comparison, Integral, Ratio** and **Root Tests**, usually by comparing with known convergent or divergent series, like the geometric or harmonic series. In this theory, it means that we establish first the convergence or divergence of a series, then establish its exact sum (when possible), or give estimates for it.

This finishes the first half of the block. We now move to deepen our understanding of the first 12 lectures.

Chapter 5: Additional, Deeper Results on Sequences and Series (2 Lectures)

In this chapter we use a more **general definition of limit of sequences** that works for any real sequence: the (ϵ, N)-definition. In lecture thirteen you shall learn how to use it, and show that it is equal to our previous definition in the context of Chapter 2. In lectures fourteen and fifteen, we use the more subtle definition to study the convergence of **alternating series**, series of real numbers that converge but, with a divergent series of the modulus of the terms. We shall also see (mainly in exercises) that such series behave in an unexpected manner.

Chapter 6: Approximation with Taylor Polynomials (3 Lectures)

Chapter 5 is entirely devoted to the estimate of errors made by replacing functions by their Taylor polynomials. In lecture sixteen we state and prove Taylor's Theorem, giving an estimate of the error term $R_n^a f$ with greater precision than in Chapter 1, so that we can estimate other error terms, in particular in lecture eighteen, where we discuss approximation of integrals. All those estimates lead to the next, and final, chapter.

Chapter 7: Power and Taylor Series (4 Lectures)

This final chapter is about functions defined as **series** (with an infinite sum depending on powers of the variable x, like $\sum_{n=0}^{\infty} \frac{x^n}{n!}$). This series happens to converge to e^x and its finite initial segments correspond to the Taylor polynomials $T_n e^x$ of e^x. From the other side, as we let n tend to ∞, what happens to $T_n^a f(x)$? How far is it from $f(x)$? Those questions are studied in the last 5 lectures. In lectures twenty one and twenty two, we derive fundamental properties shared by all **power series**. We finish in the last two lectures by applying those results to Taylor series.

Chapter 1

Taylor Polynomials

1.1 Lecture 1: Taylor Polynomials, Definition

We start with an example.

Example. What we can do to determine the behaviour of

$$p(x) = 1 + x + x^2 + x^3$$

around $x = 3$, is to recast $p(x)$ such that the behaviour of p near $x = 3$ is more transparent. Set $x = 3 + y$, $y = x - 3$, and calculate:

$$\begin{aligned} 1 + x + x^2 + x^3 &= 1 + (3 + y) + (3 + y)^2 + (3 + y)^3 \\ &= 40 + 34y + 10y^2 + y^3 \\ &= 40 + 34(x - 3) + 10(x - 3)^2 + (x - 3)^3. \end{aligned} \tag{1.1}$$

This last expression allows us to see more clearly what happens when we are **close** to $x = 3$, that is, when $|y| = |x - 3| < 1$, because then the powers of $x - 3$ decrease in value.

There is a question: the final form (1.1) of p involved some calculations (that we skipped). **How is it possible to avoid them?** The answer is to look at the **derivatives of p at $x = 3$**. We have

$$f'(x) = 1 + 2x + 3x^2, \quad f''(x) = 2 + 6x \quad \text{and} \quad f'''(x) = 6.$$

Therefore, $f(3) = 40$, $f'(3) = 34$, $f''(3) = 20$, $f'''(3) = 6$, and thus

$$p(x) = 40 + 34(x - 3) + \frac{20}{2!}(x - 3)^2 + \frac{6}{3!}(x - 3)^3 = 40 + 34(x - 3) + 10(x - 3)^2 + (x - 3)^3.$$

Both sides in the above equation are third degree polynomials, and their derivatives of order 0, 1, 2 and 3 are the same at $x = 3$, so they must be the same polynomial.

We shall generalise that idea to any function $f : [a, b] \to \mathbb{R}$. Later in MA2712 and MA2715, you shall see that those ideas can also be generalised to any function of several variables.

1.1.1 Reminder from Level 1 about Differentiable Functions

But first some reminder of Level 1 material about calculus, in particular differentiable functions. A real valued function f in one dimension is a triple $f : I \subseteq \mathbb{R} \to J \subseteq \mathbb{R}$ where

1. I is a subset of $\mathbb{R}$, called the **domain** of f, often denoted by $\boldsymbol{D(f)}$,
2. J is a subset of $\mathbb{R}$, called the **co-domain** of f,
3. $f(x)$ is the **rule** defining **the value** $f(x) \in J$ for every $x \in I$.

Remark 1.1. *When the domain and co-domains are clear from the context, we often use only the rule $f(x)$ to define $f : I \to J$, $x \mapsto f(x)$. An example is $f(x) = \sin(x)$ or $f(x) = e^x$ when $I = J = \mathbb{R}$ (for instance). This is not precise, but often convenient.*

The first, second or third derivatives of f are denoted by f', f'' or f'''. For higher derivatives, we use the notation $\boldsymbol{f^{(k)}}$ to mean the k-th derivative of f. By definition, $\boldsymbol{f^{(0)} = f}$. For $k \in \mathbb{N}$, the notation $\boldsymbol{k!}$ means ***k*-factorial**, which by definition is

$$k! = 1 \cdot 2 \ldots \cdot (k-1) \cdot k = \Pi_{j=1}^{k} j.$$

Recall that we had also defined $\mathbf{0! = 1! = 1}$. The following two theorems are fundamental.

Theorem 1.2 (Rolle's[1] Theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) such that $f(a) = f(b) = 0$, then there exists $a < c < b$ such that $f'(c) = 0$.*

The theoretical underpinning for these facts about Taylor polynomials is the Mean Value Theorem, which itself depends upon some fairly subtle properties of the real numbers.

Theorem 1.3 (Mean Value Theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). For every $x \in (a, b)$, there exists $a < \xi < x$, ξ dependent on x, such that*

$$f'(\xi) = \frac{f(x) - f(a)}{x - a}.$$

1.1.2 Definition of Taylor Polynomials

Suppose you need to do some computation with a complicated function f, and suppose that the only values of x you care about are close to some constant $x = a$. Since polynomials are simpler than most other functions, you could then look for a polynomial p which somehow 'matches' your function f for values of x close to a. And you could then replace your function f with the polynomial p, hoping that the error you make is not too big.

Which polynomial you will choose depends on when you think a polynomial 'matches' a function. In this chapter we will say that a **polynomial $\boldsymbol{p}$ of degree $\boldsymbol{n}$ matches a function $\boldsymbol{f}$ at $\boldsymbol{x = a}$** if p has the same value and the same derivatives of order 1, 2, ..., n at $x = a$ as the function f. The polynomial which matches a given function at some point $x = a$ is given in the following formula.

[1] M. Rolle (1652-1719) was a French mathematician. Largely self-taught, he worked on analysis, geometry and arithmetic. He is best remembered for the theorem bearing his name. He also invented the notation $\sqrt[n]{x}$ and adopted the notion that if $a > b$ then $-b > -a$ in opposition to various senior mathematicians of his time.

Definition 1.4. *Let $I \subseteq \mathbb{R}$ be an open interval, $f : I \to \mathbb{R}$ be a n-times differentiable function at $a \in I$. The* **Taylor polynomial $T_n^a f$ of degree n at a point a** *of f is the polynomial*

$$T_n^a f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n. \tag{1.2}$$

In the following lemma we show that $T_n^a f$ is indeed the polynomial we seek.

Theorem 1.5. *Let $I \subseteq \mathbb{R}$ be an open interval, $f : I \to \mathbb{R}$ be a n-times differentiable function at $a \in I$. The Taylor polynomial $T_n^a f$ is the* **only** *polynomial p of degree n for which*

$$p(a) = f(a), \quad p'(a) = f'(a), \quad p''(a) = f''(a), \quad \ldots, \quad p^{(n)}(a) = f^{(n)}(a)$$

holds.

Proof. The result depends from a lemma about polynomial that will be proved in the last exercise in Sheet 1a. Namely, given any polynomial p of degree n, say, and $a \in \mathbb{R}$, there exist $a_i \in \mathbb{R}$, $0 \le i \le n$, such that

$$p(x) = \sum_{i=0}^{n} a_i (x-a)^i = a_0 + a_1(x-a) + \ldots + a_n(x-a)^n. \tag{1.3}$$

Clearly, taking the derivatives of p in the shape of (1.3) and evaluating at $x = a$, we find

$$p(a) = a_0, \quad p'(0) = a_1, \quad p''(a) = 2a_2, \quad p^{(3)}(a) = 2 \cdot 3\, a_3 \quad \ldots$$

and, in general,

$$p^{(k)}(a) = k!\, a_k, \quad 0 \le k \le n.$$

Therefore, p is unique and its coefficients in (1.3) are given by

$$a_k = \frac{f^{(k)}(a)}{k!}, \quad 0 \le k \le n.$$

□

Remark 1.6. *Note that the zeroth order Taylor polynomial is just a constant,*

$$T_0^a f(x) = f(a),$$

while the first order Taylor polynomial is

$$T_1^a f(x) = f(a) + f'(a)(x-a).$$

This is exactly the **linear approximation** *of f for x close to a which was derived in Level 1.* **Taylor polynomial generalizes this first order approximation by providing 'higher order approximations' to f.**

The particular case when $a = 0$ is an important one, so it has a special name.

Definition 1.7. *When $a = 0$, $T_n^0 f$ is called the* **Maclaurin[2] polynomial of degree n** *of f, and is denoted by* **$T_n f$** *(for simplicity).*

[2] C. Maclaurin (1698-1746) was a Scottish mathematician. He got his degree at 14 years of age. He spread Newton's theory of fluxions (Newton's version of calculus), developed the integral test for series and worked on geometry. He is the founder of actuarial studies.

1.2 Lectures 2 and 3: Taylor Polynomials, Examples

We look here at a few examples of Taylor and Maclaurin polynomials.

1.2.1 Example: Compute and plot $T_n f$ for $f(x) = e^x$

All the derivatives of f are identical, $f(x) = f^{(n)}(x)$, $\forall n \in \mathbb{N}$, so that $f^{(n)}(0) = 1$. Therefore the first three Taylor polynomials of e^x at $a = 0$ (or simply Maclaurin polynomials of e^x) are

$$T_0 f(x) = 1,$$
$$T_1 f(x) = 1 + x,$$
$$T_2 f(x) = 1 + x + \frac{1}{2}x^2.$$

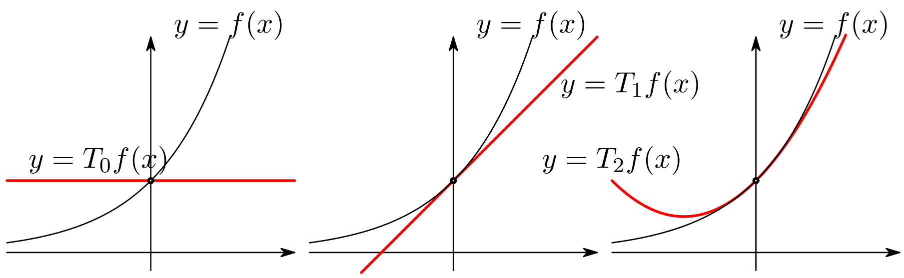


Figure 1.1: The Maclaurin polynomials of degree 0, 1 and 2 of e^x

Contemplating the graphs in Figure 1.1, the following comments are of order:

1. The zeroth order Taylor polynomial $T_0 f$ has the right value at $x = 0$ but it does not know whether or not the function f is increasing at $x = 0$. $T_0 f(x)$ is close to e^x for small x, by virtue of the continuity of e^x, because e^x does not change very much if x stays close to $x = 0$.

2. $T_1 f(x) = 1 + x$ corresponds to the tangent line to the graph of $y = e^x$ at $x = 0$, and so it also captures the fact that the function f is increasing near $x = 0$, but it does not see if the graph of f is curved up or down at $x = 0$. Clearly $T_1 f(x)$ is a better approximation to e^x than $T_0 f(x)$.

3. The graphs of both $T_0 f$ and $T_1 f$ are straight lines, while the graph of $y = e^x$ is curved (in fact, convex). The graph of $T_2 f$ is a parabola and, since it has the same first and second derivative at $x = 0$, it captures this convexity. It has the right curvature at $x = 0$. So it seems that

$$y = T_2 f(x) = 1 + x + x^2/2$$

is an approximation to $y = e^x$ which beats both $T_0 f$ and $T_1 f$.

In Figure 1.2, the top edge of the shaded region is the graph of $y = e^x$. The graphs are of the functions $y = 1 + x + Cx^2$ for various values of C. These graphs all are tangent at $x = 0$, but one of the parabolas matches the graph of $y = e^x$ better than any of the others.

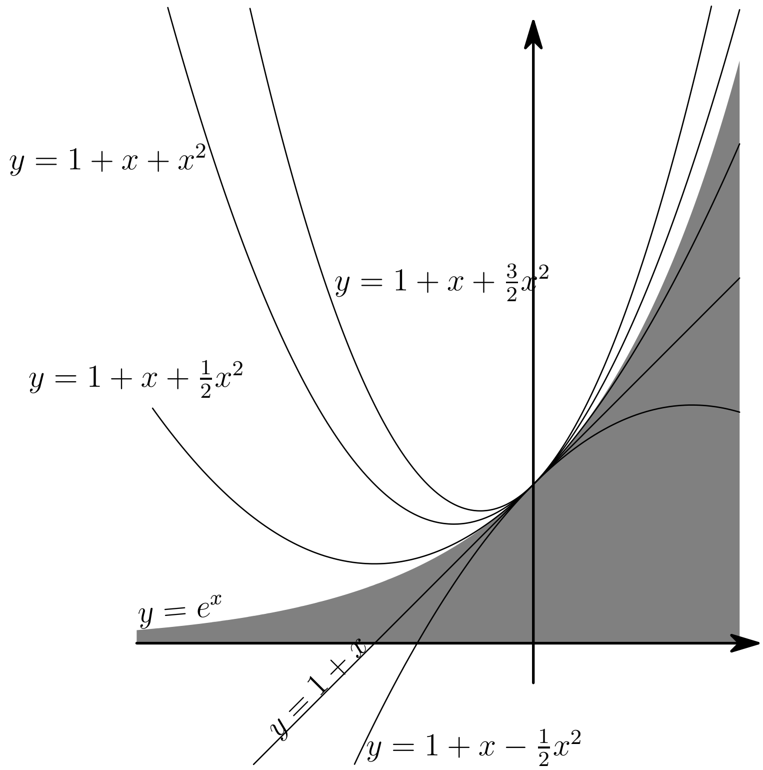


Figure 1.2: Maclaurin polynomials of e^x and perturbations

1.2.2 Example: Find the Maclaurin polynomials of $f(x) = \sin x$

When you start computing the derivatives of $\sin x$ you find

$$f(x) = \sin x, \quad f'(x) = \cos x, \quad f''(x) = -\sin x, \quad f^{(3)}(x) = -\cos x, \tag{1.4}$$

and thus

$$f^{(4)}(x) = \sin x.$$

So, after four derivatives, you are back to where you started, and the sequence of derivatives of $\sin x$ cycles through the pattern (1.4). At $x = 0$, you then get the following values for the derivatives

$$f^{(4j)}(0) = f^{(4j)}(0) = 0, \quad f^{(4j+1)}(0) = -f^{(4j+3)}(0) = 1, \quad j \in \mathbb{N}.$$

This gives the following Maclaurin polynomials

$$\begin{aligned}
T_0 f(x) &= 0, \\
T_1 f(x) &= x, \\
T_2 f(x) &= T_1 f(x), \\
T_3 f(x) &= x - \frac{x^3}{3!} = T_4 f(x), \\
T_5 f(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} = T_6 f(x) \ldots
\end{aligned}$$

Note that the Maclaurin polynomial $T_n f$ of any function is a polynomial of degree at most n, and this example shows that the degree can sometimes be strictly less.

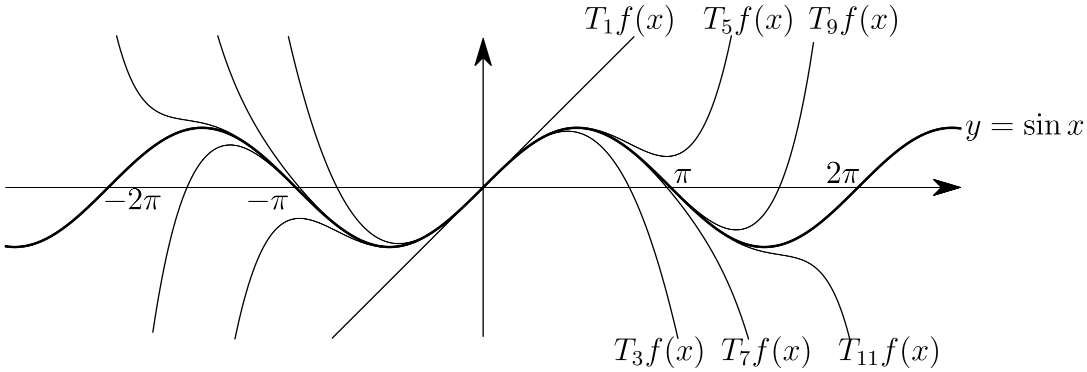


Figure 1.3: Maclaurin polynomials of $\sin x$

1.2.3 Find the Maclaurin polynomial $T_{11}f$ for $f(x) = \sin(x^2)$

Note, if we tried to do this in the same way as our other examples it would be lengthy. For the first derivative we would need the Chain Rule, after that we would need the Product Rule and we would get two terms. After that we would need the Chain and Product Rules again and we would need three terms (though, after some algebra, it combined into two again). After that, it just keeps getting worse, though, most of the terms vanish at $x = 0$. We would find that

$$T_{11}f(x) = x^2 - \frac{1}{3!}x^6 + \frac{1}{5!}x^{10},$$

basically replacing x by x^2 into the Maclaurin polynomial of degree 5 of $\sin(x)$:

$$T_{11}f(x) = T_{10}f(x) = T_5 \sin x(x^2).$$

We shall see in Lecture 1.2 that this is to be expected and will be a useful way not to have to calculate an enormous number of derivatives.

1.2.4 Questions for Chapter 6: Error Estimates

In this subsection we hint at some issues we have ignored but shall come back to in Chapter 6 when we will have the technical tools to approach it.

We have seen how $T_n^a f$ approximate f and its derivatives at $x = a$. What about in an interval around $x = a$? Many mathematical operations need the values of a function on an interval, not only at one given point $x = a$. Having a polynomial approximation that works **all along an interval** is much more substantive than an evaluation at a single point. The Taylor polynomial $T_n^a f(x)$ is almost never exactly equal to $f(x)$, but often it is a good approximation, especially if $|x - a|$ is small.

Given an interval I around a, a tolerance $\epsilon > 0$ and the order n of the approximation, here are the big questions:

1. Given $T_n^a f$. **Within what tolerance** does $T_n^a f$ approximate f on I?
2. Given $T_n^a f$ and ϵ. **On how large an interval** $I \ni a$ does $T_n^a f$ achieve that tolerance?
3. Given f, $x = a$, $I \ni a$ and ϵ. Find **how many terms** n must be used in $T_n^a f$ to approximate f to within ϵ on I.

To be able to be precise about those statements we shall introduce the following idea.

Definition 1.8. *Let $I \subseteq \mathbb{R}$ be an open interval, $a \in I$ and $f : I \to \mathbb{R}$ be a n-times differentiable function, then*

$$\boldsymbol{R_n^a f(x) = f(x) - T_n^a f(x)} \tag{1.5}$$

is called the **n-th order remainder (or error) term** *of the Taylor polynomial $T_n^a f$ of f.*

In general we shall see that there are formula for the n-th order remainder we can estimate to give answers to the previous questions. But this is for the future. At present we concentrate to the calculations and some use of the Taylor polynomials without detailed justification. They are for a revisit later.

Proposition 1.9 (Peano[3]'s Form)**.** *Let $I \subseteq \mathbb{R}$ be an open interval, $a \in I$ and $f : I \to \mathbb{R}$ be a n-times differentiable function, then there exists a neighbourhood of $a \in J \subseteq \mathcal{I}$ and a function $h_n : J \to \mathbb{R}$ such that the error term $R_n^a f$ satisfies*

$$R_n^a f(x) = h_n(x)\,(x-a)^n \tag{1.6}$$

with $\lim_{x\to a} h_n(x) = 0$.

Proof. The proof is not really difficult. We leave it to the exercise sheet. It boils down to using l'Hôpital Rule n-times on

$$h_n(x) = \frac{R_n^a f(x)}{(x-a)^n}, \quad x \neq a.$$

□

[3] G. Peano (1858-1932) was an Italian analyst and a founder of mathematical logic. He worked on differential equations, set theory and axiomatic. Peano had a great skill in seeing that theorems were incorrect by spotting exceptions. At the end of his life he became interested in universal languages, both between humans and for teaching and working in mathematics.

1.3 Lecture 4 and 5: Calculus of Taylor Polynomials

The obvious question to ask about Taylor polynomials is 'What are the first so-many terms in the Taylor polynomial of some function expanded at some point?'. The most straightforward way to deal with this is just to do what is indicated by the formula (1.2): take however high order derivatives you need and plug in $x = a$. However, very often this is not at all the most efficient method. Especially in a situation where we are interested in a composite function f of the form $f(x^n)$ or, more generally, $f(p(x))$, where p is a polynomial, even $f(g(x))$, where g is a function, all with 'familiar' Taylor polynomial expansions.

The fundamental principles of those calculations follow from the following results. Recall that the **degree of a polynomial** is the degree of the highest **non zero** monomial appearing in p. It is denoted $\mathbf{deg}(\boldsymbol{p})$.

Lemma 1.10. ***1.*** *Let p and q two polynomials. Then $\deg(p \cdot q) = \deg(p) + \deg(q)$ and $\deg(p(q)) = \deg(p) \cdot \deg(q)$.*

2. *When the Taylor polynomial $T_n^a f$ of a function f is in the form (1.2), we obtain* **easily** *$T_k^a f$, $k \leq n$, by ignoring the terms of powers $(x-a)^l$, with $l > k$, in $T_n^a f$.*

The proof of this lemma will be provided in Chapter 6, but you can attempt it, it is not difficult.

1.3.1 General Results

Using Lemma 1.10, we can show how to get the Taylor polynomials of composed functions.

Theorem 1.11. ***1.*** *The Taylor polynomial of the sum h of two functions f and g is given by the sum of their Taylor polynomials:*

$$T_n^a h = T_n^a f + T_n^a g. \tag{1.7}$$

2. *The Taylor polynomial of the product of two functions f and g is given by the terms of degree up to n of the product of their Taylor polynomials. Namely, the terms of degree up to n of:*

$$T_n^a h = T_n^a (T_n^a f \cdot T_n^a g). \tag{1.8}$$

Examples.

1. Consider $f(x) = e^x$ and $g(x) = \sin(x)$. To illustrate (1.7) and (1.8), we calculate the Maclaurin polynomials of order 3 of $h_1(x) = f(x) + g(x)$ and of $h_2(x) = e^x \sin(x)$. For h_1,

$$T_3 h_1(x) = T_3 f(x) + T_3 g(x) = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{3!}\right) + \left(x - \frac{x^3}{3!}\right) = 1 + 2x + \frac{x^2}{2}.$$

The idea from (1.8) is just that you multiply the polynomials for e^x and $\sin(x)$. And so,

$$T_3 f(x) = T_3 \left((1 + x + \frac{x^2}{2} + \frac{x^3}{3!})(x - \frac{x^3}{3!})\right) = x + x^2 + (-\frac{1}{3!} + \frac{1}{2})x^3 = x + x^2 + \frac{1}{3}x^3. \tag{1.9}$$

2. As a more complicated example of (1.8), let $f(x) = e^{2x}/(1+3x)$. We exploit our knowledge of the Maclaurin polynomial of both factors e^{2x} and $1/(1+x)$. As an example, what is $T_3 f$? Let

$$T_3 e^{2x} = 1 + 2x + \frac{2^2 x^2}{2!} + \frac{2^3 x^3}{3!} = 1 + 2x + 2x^2 + \frac{4}{3}x^3,$$

and

$$T_3(\frac{1}{1+3x}) = 1 - 3x + 9x^2 - 27x^3.$$

Then multiply these two

$$T_3 f(x) = 1 - x + 5x^2 - \frac{41}{3}x^3.$$

The composition of functions is more subtle.

Theorem 1.12. *The Taylor polynomial of the composition h of two functions f and g, $h(x) = f(g(x))$ is given by the terms of degree up to n of the composition of their Taylor polynomials. Namely,*

$$T_n^a h = T_n^a \left(T_n^{g(a)} f(T_n^a g)\right). \tag{1.10}$$

Remark 1.13. *Pay attention to the fact that, with the composition of function, we need the Taylor polynomial of the second function f* **about the value of the first one $g(a)$**. *In the expression $h(x) = f(g(x))$ we apply first g then f. Remark that we often use, in Theorem 1.12, functions g such that $g(0) = 0$ so that (1.10) becomes simply*

$$T_n h = T_n \left(T_n f(T_n g)\right). \tag{1.11}$$

Example. As a first example of (1.10), let $f(x) = 1/(1+x^2)$. What is $T_{12} f$?

The direct approach is how NOT to compute it. Diligently computing derivatives one by one, you find

$$\begin{aligned}
f(x) &= \frac{1}{1+x^2} && \text{so } f(0) = 1,\\
f'(x) &= \frac{-2x}{(1+x^2)^2} && \text{so } f'(0) = 0,\\
f''(x) &= \frac{6x^2-2}{(1+x^2)^3} && \text{so } f''(0) = -2,\\
f^{(3)}(x) &= 24\frac{x-x^3}{(1+x^2)^4} && \text{so } f^{(3)}(0) = 0,\\
f^{(4)}(x) &= 24\frac{1-10x^2+5x^4}{(1+x^2)^5} && \text{so } f^{(4)}(0) = 24 = 4!,\\
f^{(5)}(x) &= 240\frac{-3x+10x^3-3x^5}{(1+x^2)^6} && \text{so } f^{(4)}(0) = 0,\\
f^{(6)}(x) &= -720\frac{-1+21x^2-35x^4+7x^6}{(1+x^2)^7} && \text{so } f^{(4)}(0) = 720 = 6!.\\
&\vdots
\end{aligned}$$

I am getting tired of differentiating - can you find $f^{(12)}(x)$? After a lot of work we give up at the sixth derivative, and all we have found is

$$T_6 f(x) = 1 - x^2 + x^4 - x^6.$$

By the way,

$$f^{(12)}(x) = 479001600\, \frac{1 - 78\,x^2 + 715\,x^4 - 1716\,x^6 + 1287\,x^8 - 286\,x^{10} + 13\,x^{12}}{(1+x^2)^{13}}$$

and $479001600 = 12!$.

The right approach to finding $T_{12}f$ is to use (1.10). We have seen that if $g(t) = 1/(1-t)$ then

$$T_6 g(t) = 1 + t + t^2 + t^3 + t^4 + t^6.$$

Following (1.10), because $t = 0$ when $x = 0$, substitute $t = -x^2$ in this limit, we get

$$T_{12} f(x) = 1 - x^2 + x^4 - x^6 + \cdots + x^{12}.$$

Note that, in general, we get easily $T_{2n}f = T_{2n+1}f$ from $T_n g$ as

$$T_{2n} f(x) = T_n g(-x^2) = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n}. \tag{1.12}$$

A more complicated one is (1.13) in the next section.

Example. To find the Maclaurin polynomial $T_4 f$ for $f(x) = \exp(\sin(x))$, we use

$$T_4 \sin(x) = x - \frac{x^3}{6}$$

and

$$T_4 \exp(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}.$$

Note that because $\sin(0) = 0$, $T_4^{\sin(0)} \exp(x) = T_4^0 \exp(x) = T_4 \exp(x)$. And so, from (1.10),

$$\begin{aligned} T_4 f(x) &= 1 + (x - x^3/6) + (x - x^3/6)^2/2 + x^3/6 + x^4/24 \\ &= 1 + x + \frac{x^2}{2} - \frac{x^4}{8}. \end{aligned} \tag{1.13}$$

Differentiation and integration are straightforward.

Theorem 1.14. *1. The Taylor polynomial of the derivative of f is given by the derivative term per term of the Taylor polynomial, that is,*

$$T_n^a f'(x) = \left(T_{n+1}^a f(x)\right)'. \tag{1.14}$$

2. *The Taylor polynomial of the anti-derivative (primitive/integral) F of f is given by integrating term per term the Taylor polynomial, that is,*

$$T^a_{n+1}F(x) = C + \int T^a_n f(x)\, dx. \tag{1.15}$$

Example. To find the Maclaurin polynomials for $f(x) = \tan^{-1}(x) = \arctan(x)$, we start with an example we already know. Thinking about derivatives and anti-derivatives, we see that f is the anti-derivative of $f'(x) = \dfrac{1}{1+x^2}$. So, we would have a plan if we knew the Maclaurin polynomial for $\dfrac{1}{1+x^2}$. Well, we know it. So the plan is this:

$$T_{2n}f'(x) = 1 - x^2 + x^4 - x^6 + x^8 + \ldots + (-1)^n x^{2n}.$$

And so, using (1.14) and (1.15),

$$\begin{aligned} T_{2n+1}f(x) &= \int \left(1 - x^2 + x^4 - x^6 + x^8 + \ldots + (-1)^n x^{2n}\right) dx \\ &= C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1}. \end{aligned}$$

Because $f(0) = \arctan(0) = 0$, $C = 0$ and so

$$T_{2n+1}f(x) = \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{2k+1}. \tag{1.16}$$

When $a = 0$, Theorems 1.11, 1.12 and 1.14 lead to the following classical, useful examples about Maclaurin polynomials.

Corollary 1.15. *Let p be a polynomial* **such that $p(0) = 0$.**

1. *If $h(x) = \exp(p(x))$, then*

$$T_n h(x) = T_n \left(\sum_{k=0}^{n} \frac{1}{k!} (p(x))^k \right).$$

2. *If $h(x) = \dfrac{1}{1-p(x)}$, then*

$$T_n h(x) = T_n \left(\sum_{k=0}^{n} (p(x))^k \right).$$

3. *If $h(x) = \sin(p(x))$, then*

$$T_n h(x) = T_n \left(\sum_{k=0}^{n} (-1)^k \frac{1}{(2k+1)!} (p(x))^{2k+1} \right).$$

4. *If* $h(x) = \cos(p(x))$, *then*

$$T_n h(x) = T_n \left(\sum_{k=0}^{n} (-1)^k \frac{1}{(2k)!} (p(x))^{2k} \right).$$

5. *If* $h(x) = \ln(1 + p(x))$, *then*

$$T_n h(x) = T_n \left(\sum_{k=1}^{n} (-1)^{k+1} \frac{1}{k} (p(x))^{k} \right).$$

We shall illustrate these results in the next few lectures.

1.4 Lecture 6: Various Applications of Taylor Polynomials

We shall look at two areas of use of Taylor approximations:

1. generalised criteria using derivatives to determine the type of a critical point (maximum, minimum or saddle points),
2. generalised l'Hôpital[4] Rule for limits.

1.4.1 Relative Extrema

At Level 1 you have seen that if a function $f : I \to \mathbb{R}$ has a critical point at $a \in I$, that is, if $f'(a) = 0$, then f has

1. a **(local) maximum** at $x = a$ if $f''(a) < 0$,
2. a **(local) minimum** at $x = a$ if $f''(a) > 0$,
3. an **degenerate critical point** at $x = a$ if $f''(a) = 0$.

In that last case, what can we say using Taylor expansions of f at $x = a$?

Theorem 1.16 (Relative Extrema)**.** *Let $I \subseteq \mathbb{R}$ be an open set, $a \in I$, and $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be n-times continuously differentiable at $x = a$, $n \geq 2$. Suppose*

$$f'(a) = \ldots = f^{(n-1)}(a) = 0, \quad f^{(n)}(a) \neq 0.$$

Then,

1. *if n is odd, a is not a (local) extrema, it is a saddle point,*
2. *if n is even,*
   (i) a is a local maximum if $f^{(n)}(a) < 0$,
   (ii) a is a local minimum if $f^{(n)}(a) > 0$.

Proof. For x close to a, we have

$$\begin{aligned} f(x) &= T_n^a f(x) + R_n^a f(x) = f(a) + f'(a)(x-a) + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n^a f(x) \\ &= f(a) + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n^a f(x) \\ &= f(a) + \left(\frac{f^{(n)}(a)}{n!} + h_n(x)\right)(x-a)^n, \end{aligned}$$

[4] G.F.A. marquis de l'Hôpital (1661-1704) was a French mathematician. He wrote the first textbook of Calculus where he gave the rule having his name.

where h_n appears in the Peano form of the remainder.

Define $g_1(x) = \dfrac{f^{(n)}(a)}{n!} - h_n(x)$, and denote by g_2 the constant function $g_2(x) = \dfrac{f^{(n)}(a)}{n!}$.

The function $g_1 = g_1(x)$ is close to the constant function $g_2(x) = \dfrac{f^{(n)}(a)}{n!}$ when x is close to a. Therefore, we can deduce that for x close to a the value $g_1(x)$ has the same sign as the constant $\dfrac{f^{(n)}(a)}{n!}$.

Thus, it is enough to study the local extrema of the approximation

$$f(a) + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

It is now simple calculus to show that our conclusions are correct. □

Examples.

1. Let $f(x) = \sin(x^2) - x^2\cos(x)$. Determine the type of the critical point $x = 0$.

The first non zero terms of the Maclaurin polynomials are

$$\begin{aligned}
\sin(x^2) &= x^2 - \frac{x^6}{6} + \ldots, \\
x^2\cos(x) &= x^2\left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right) = x^2 - \frac{x^4}{2} + \frac{x^6}{24} + \ldots.
\end{aligned}$$

And so, $f(x) = \dfrac{x^4}{2} - \dfrac{5}{24}x^6 + \ldots$. Hence, $x = 0$ is a (local) minimum.

2. Let $g(x) = \cos(x^2) - e^{\sin x} - \ln(1-x) + ax^3$ where $a \in \mathbb{R}$ is a parameter. Discuss the type of the critical point $x = 0$ as a function of a.

The first non zero terms of the Maclaurin polynomials are

$$\begin{aligned}
\cos(x^2) &= 1 - \frac{x^4}{2} + \frac{x^8}{24} + \ldots, \\
e^{\sin(x)} &= 1 + \left(x - \frac{x^3}{6}\right) + \frac{1}{2}\left(x - \frac{x^3}{6}\right)^2 + \frac{1}{6}\left(x - \frac{x^3}{6}\right)^3 + \ldots \\
&= 1 + x + \frac{x^2}{2} - \frac{x^4}{8} + \ldots, \\
-\ln(1-x) &= x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots.
\end{aligned}$$

And so,

$$g(x) = \left(\frac{1}{3} + a\right)x^3 - \frac{x^4}{8} + \ldots.$$

Therefore, when $a \neq -1/3$, $x = 0$ is a saddle point, but when $a = -1/3$, $x = 0$ is a (local) maximum. This local maximum is indeed 'illustrated' in Figure 1.4.

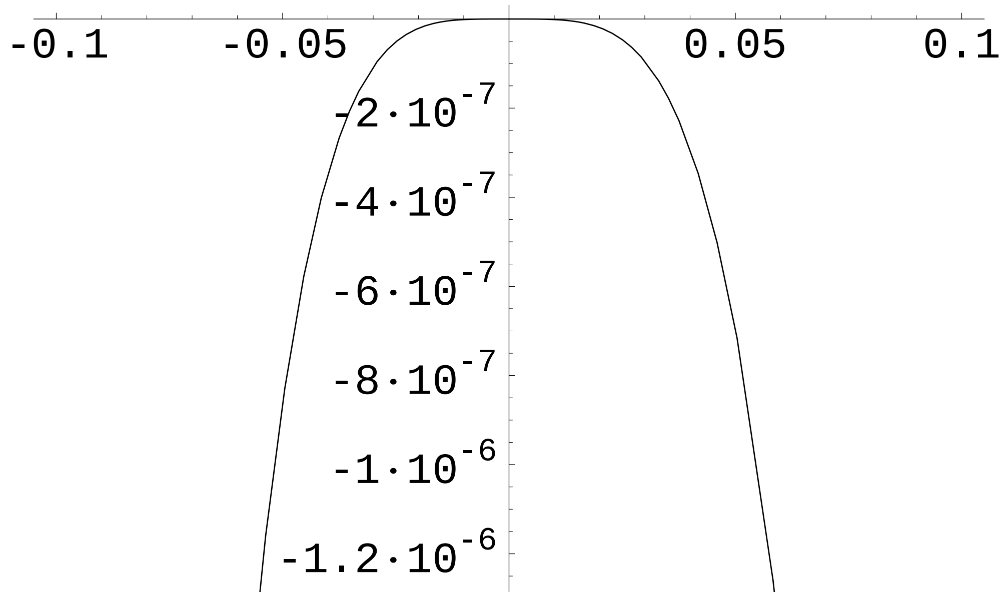


Figure 1.4: The graph near $x = 0$ of $g(x) = \cos(x^2) - e^{\sin x} - \ln(1-x) - \dfrac{x^3}{3}$.

1.4.2 Limits

At Level 1 you have seen l'Hôpital's Rule. Let $h(x) = \dfrac{f(x)}{g(x)}$ and suppose that there exists an a such that $f(a) = g(a) = 0$ and $g'(a) \neq 0$, then $\lim\limits_{x\to a} h(x) = \dfrac{f'(a)}{g'(a)}$. But if $g'(a) = 0$ this is of no help. We now prove an extension of that result using Taylor polynomials.

Theorem 1.17 (Generalised l'Hôpital Rule)**.** *Let $I \subseteq \mathbb{R}$ be an open set, $a \in I$, and $f, g : I \subseteq \mathbb{R} \to \mathbb{R}$ be $n+1$, resp. $m+1$,-times continuously differentiable at $x = a$. Suppose*

$$\begin{aligned} 0 &= f(a) = f'(a) = \ldots = f^{(n-1)}(a) = 0, \quad f^{(n)}(a) \neq 0, \\ 0 &= g(a) = g'(a) = \ldots = g^{(m-1)}(a) = 0, \quad g^{(m)}(a) \neq 0. \end{aligned}$$

Then, $h : I \to \mathbb{R}$, defined by $h(x) = \dfrac{f(x)}{g(x)}$, has the following limits as $x \to a$:

1. *if $n > m$, $\lim_{x\to a} h(x) = 0$,*
2. *if $m = n$, $\lim\limits_{x\to a} h(x) = \dfrac{f^{(n)}(a)}{g^{(n)}(a)}$,*
3. *if $n < m$, the limit $\lim_{x\to a} h(x)$ is unbounded.*

Proof. For x close to a,

$$\begin{aligned} f(x) &= f(a) + f'(a)(x-a) + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n^a f(x), \\ &= \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n^a f(x) = \left(\frac{f^{(n)}(a)}{n!} + h_n(x)\right)(x-a)^n, \end{aligned}$$

where h_n appears in the Peano's form of the remainder. Similarly for g. For x close to a,

$$\begin{aligned} g(x) &= g(a) + g'(a)(x-a) + \ldots + \frac{g^{(m)}(a)}{m!}(x-a)^m + R^a_m g(x), \\ &= \frac{g^{(m)}(a)}{m!}(x-a)^m + R^a_m g(x) = \left(\frac{g^{(m)}(a)}{m!} + h_m(x)\right)(x-a)^m. \end{aligned}$$

Now, define

$$h(x) = \frac{f(x)}{g(x)} = (x-a)^{n-m}\left(\frac{\frac{f^{(n)}(a)}{n!} + h_n(x)}{\frac{g^{(m)}(a)}{m!} + h_m(x)}\right).$$

Note that the big bracket is bounded when $x \to a$. Now we consider our three cases:

1. when $n > m$, clearly $\lim_{x\to a} h(x) = 0$;

2. when $n = m$, h simplifies into

$$h(x) = \frac{\frac{f^{(n)}(a)}{n!} + h_n(x)}{\frac{g^{(n)}(a)}{n!} + h_n(x)},$$

and the conclusion about the limit follows;

3. when $n < m$, clearly $\lim_{x\to a} h(x) = \pm\infty$ because $(x-a)^{n-m}$ will blow-up.

□

Example. Discuss the limit as $x \to 0$ of

$$h(x) = \frac{\sin(x^2) - x^2\cos(x)}{\cos(x^2) - e^{\sin x} + \ln(1-x) + ax^3}$$

as a function of a. Using the previous results, we get

$$\lim_{x\to 0} h(x) = \lim_{x\to 0} \frac{x^4/2}{(1/3+a)x^3 - x^4/8} = \begin{cases} 0, & a \neq -1/3; \\ -4, & a = -1/3. \end{cases}$$

Remark 1.18. *In Chapter 6, with a more precise expression for the error term, we shall look also at the use of Taylor polynomials approximations to estimate integral, for instance*

$$\int_a^b f(x)\,dx \approx \int_a^b T^a_n f(x)\,dx \approx \int_a^b T^b_n f(x)\,dx \approx \int_a^b T^c_n f(x)\,dx,$$

where $c = \frac{a+b}{2}$*. Note that those are only* **approximations***. In Chapter 6 we shall be able to estimate the maximum error of the different formulas and so determine which one is more appropriate depending on what we want to achieve.*

1.4.3 How to Calculate Complicated Taylor Polynomials?

We consider a few examples illustrating Corollary 1.15 and showing how to handle the figuring out of Maclaurin polynomials of arbitrary orders (when such formulae exists). The question is to find the Maclaurin polynomial of order n of the following functions.

Examples.

1. Let $f(x) = \dfrac{x+2}{3+2x}$. Simplify

$$f(x) = \frac{(x+2)}{3} \, \frac{1}{(1+(2x)/3)}.$$

We know that

$$T_n\left(\frac{1}{(1+(2x)/3)}\right) = \sum_{k=0}^{n} (-1)^k \left(\frac{2x}{3}\right)^k.$$

Multiplying by $\dfrac{(x+2)}{3}$ and collecting the terms up to power x^n we find

$$T_n f(x) = \frac{2}{3} + \sum_{k=1}^{n} (-1)^k \frac{2^{k-1}}{3^{k+1}} \, x^k.$$

2. Let $f(x) = \dfrac{1}{\sqrt{1-x}}$. You can use directly the Binomial Theorem (from Level 1) or calculate the derivatives of f. We get

$$f^{(k)}(0) = \frac{1 \cdot 3 \cdot 5 \cdots (2k-1)}{2^k}.$$

Recall that $2 \cdot 4 \cdot 6 \cdots 2k = 2^k (k!)$. And so by multiplying by the product of even terms, top and bottom, we get $f^{(k)}(0) = \dfrac{(2k)!}{4^k (k!)}$. Therefore

$$T_n f(x) = \sum_{k=0}^{n} \frac{(2k)!}{4^k (k!)^2} \, x^k.$$

Remark 1.19. *Note that f is equal to (-2) times the derivative of $g(x) = \sqrt{1-x}$. So, if we know $T_n g$, $T_n f = -2(T_n g)'$.*

3. Let $f(x) = \sqrt{1+x^4}$. We replace $z = x^4$ into the Maclaurin polynomials of $g(z) = \sqrt{1+z}$. Recall that

$$T_n g(z) = 1 + \sum_{k=1}^{n} (-1)^{k+1} \frac{(2k)!}{4^k (k!)^2 (2k-1)} z^k.$$

And so,

$$T_n f(x) = 1 + \sum_{k=1}^{\lfloor n/4 \rfloor} (-1)^{k+1} \frac{(2k)!}{4^k (k!)^2 (2k-1)} x^{4k}.$$

4. Let $f(x) = \ln\left(\sqrt{\dfrac{1+x}{1-x}}\right)$. We simplify

$$f(x) = \frac{1}{2}\ \ln(1+x) - \frac{1}{2}\ \ln(1-x).$$

Recall the Maclaurin polynomials

$$\begin{aligned} T_n \ln(1+x) &= \sum_{k=1}^{n} (-1)^{k+1} \frac{x^k}{k}, \\ T_n \ln(1-x) &= -\sum_{k=1}^{n} \frac{x^k}{k}. \end{aligned}$$

Hence,

$$T_n f(x) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \frac{x^{2k+1}}{2k+1}.$$

5. Let $f(x) = \arctan(x)$. Recall that $f'(x) = \dfrac{1}{1+x^2}$. Use Theorem 1.14, integrating the Maclaurin polynomial of f':

$$T_n f'(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k x^{2k}.$$

We get that

$$T_n f(x) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k \frac{x^{2k+1}}{2k+1}.$$

Summary of Chapter 1

We have seen:

- the definition of a **Taylor polynomial** and the **error term**;
- the calculus of Taylor polynomials and its use to determine Taylor polynomials of complicated functions using known expansions;
- known expansions for functions like $\dfrac{1}{1-x}$;
- how to use Taylor polynomials to calculate limits and determine the type of a degenerate critical point.

1.5 Exercise Sheet 1

1.5.1 Exercise Sheet 1a

1. (i) Find a second order polynomial q such that $q(7) = 43$, $q'(7) = 19$ and $q''(7) = 11$.

(ii) Find a third order polynomial p such that $p(2) = 3$, $p'(2) = 8$, and $p''(2) = -1$. How many possibilities are there?

(iii) Find all third order polynomials p satisfying $p(0) = 1$, $p'(0) = -3$, $p''(0) = -8$ and $p'''(0) = 24$.

2. Let $f(x) = \sqrt{x + 25}$.

(i) Find the largest domain for the rule $f(x)$.

(ii) Find the polynomial $p(x)$ of degree three such that $p^{(k)}(0) = f^{(k)}(0)$ for $k = 0, 1, 2, 3$.

3. Compute $T_0^a f(x)$, $T_1^a f(x)$ and $T_2^a f(x)$ for the following functions.

(i) $f(x) = x^3$, $a = 0$; then for $a = 1$ and $a = 2$.

(ii) $f(x) = \dfrac{1}{x}$, $a = 1$. Also do $a = 2$.

(iii) $f(x) = \sqrt{x}$, $a = 1$.

(iv) $f(x) = \ln x$, $a = 1$. Also do $a = e^2$.

(v) $f(x) = \ln(\sqrt{x})$, $a = 1$.

(vi) $f(x) = \sin(2x)$, $a = 0$, also do $a = \pi/4$.

(vii) $f(x) = \cos(x)$, $a = \pi$.

(viii) $f(x) = (x - 1)^2$, $a = 0$, and also do $a = 1$.

(ix) $f(x) = \dfrac{1}{e^x}$, $a = 0$.

4. Determine the Maclaurin polynomial of order 4 for the following rules:

(i) $f(x) = \exp(x^2 + x)$.

(ii) $f(x) = \ln(\dfrac{1}{1 - x})$.

(iii) $f(x) = e^{\cos x}$.

(iv) $f(x) = e^{-x} \cos x$.

5. Find the Maclaurin polynomials $T_n f$ of **any order n** for the following rules. (i) $f(t) = e^{kt}$, for some constant k. (ii) $f(t) = e^{1+t}$. (iii) $f(t) = e^{-t^2}$. (iv) $f(t) = \cos(t^5)$. (v) $f(t) = \dfrac{1+t}{1-t}$. (vi) $f(t) = \dfrac{1}{1+2t}$. (vii) $f(t) = \dfrac{1}{2-t}$. (viii) $f(t) = \dfrac{e^t}{1-t}$. (ix) $f(t) = \dfrac{1}{\sqrt{1-t}}$. (x) $f(t) = \dfrac{1}{2-t-t^2}$ (xi) $f(t) = \ln(1 - t^2)$. (xii) $f(t) = \sin t \cos t$.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = |x|^3$.

(i) Determine whether f has Taylor polynomials of any order n at

[a] $a = 0$?
[b] $a = 1$?
[c] $a = -1$?

(ii) When f has Taylor polynomials, determine them (in the three cases).

7. Determine the following limits.

(i) $\lim_{x\to 0} \dfrac{x(1+\cos x) - 2\tan x}{2x - \sin x - \tan x}$;

(ii) $\lim_{x\to 1} \dfrac{1 - x + \ln x}{1 - \sqrt{2x - x^2}}$;

(iii) $\lim_{x\to 0} \left(\dfrac{1}{x} - \dfrac{1}{\ln(1+x)}\right)$;

(iv) $\lim_{x\to 0} \left(\dfrac{1}{x} - \dfrac{1}{(\sin x)^2}\right)$;

(v) Determine $\alpha \in \mathbb{R}$ such that the following limit is finite:

$$\lim_{x\to 0} \frac{xe^{-\sin(x)} - \sin(x) + \alpha x^2}{x - \sinh(x)},$$

and calculate that limit.

8. Determine the type of critical points for the following functions.

(i) $f(x) = \sin(x^2) - x^2\cos(x)$ at $x = 0$;

(ii) $f(x) = \sin x - \dfrac{x + ax^3}{1 + bx^2}$ at $x = 0$; discuss in terms of a and b and determine the leading order term.

9. Prove Proposition 1.9.

10. * Show that given any polynomial p of degree n, say, and $a \in \mathbb{R}$, there exist $a_i \in \mathbb{R}$, $0 \le i \le n$, such that

$$p(x) = \sum_{i=0}^{n} a_i(x-a)^i = a_0 + a_1(x-a) + \ldots + a_n(x-a)^n.$$

Short Feedback for Sheet 1a

1. Use Taylor's formula.

(i) $q(x) = \dfrac{11}{2}x^2 - 58x + \dfrac{359}{2}$.

(ii) $p(x) = x^3 - \dfrac{13}{2}x^2 + 22x - 23$. There is an infinite number determined by

$$p(x) = -\frac{1}{2}x^2 + 10x - 15 + c(x-2)^3,$$

where $c \in \mathbb{R}$.

(iii) $p(x) = 4x^3 - 4x^2 - 3x + 1$.

2. (i) $[-25, \infty)$.

(ii) $5 + \dfrac{x}{10} - \dfrac{x^2}{1000} + \dfrac{x^3}{5 \cdot 10^4}$.

3. $T_2^a f(x)$ is only given (think why?).

(i) $T_2 f(x) = 0$, $T_2^1 f(x) = 1+3(x-1)+3(x-1)^2$ and $T_2^2 f(x) = 8+12(x-2)+6(x-2)^2$.

(ii) $T_2^1 f(x) = 1 - (x-1) + (x-1)^2$ and $T_2^2 f(x) = \frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{8}(x-2)^2$.

(iii) $T_2^1 f(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2$.

(iv) $T_2^1 f(x) = (x-1) - \frac{1}{2}(x-1)^2$ and $T_2^{e^2} f(x) = 2 + e^{-2}(x-e^2) - \frac{e^{-4}}{2}(x-e^2)^2$.

(v) Hint: note that $\ln(\sqrt{x}) = \frac{1}{2}\ln(x)$, then use the answer of (iv)!

(vi) $T_2 f(x) = 2x$ and $T_2^{\pi/4} f(x) = 1 - 2(x-\pi/4)^2$.

(vii) $T_2^{\pi} f(x) = -1 + \frac{1}{2}(x-\pi)^2$.

(viii) $T_2 f(x) = 1 - 2x + x^2$ and $T_2^1 f(x) = (x-1)^2$.

(ix) $T_2 f(x) = 1 - x + \frac{x^2}{2}$.

4. (i) $T_4 f(x) = 1 + x + \dfrac{3}{2}x^2 + \dfrac{7}{6}x^3 + \dfrac{25}{24}x^4$.

Hint: use $T_4 e^z = 1 + z + \dfrac{z^2}{2!} + \dfrac{z^3}{3!} + \dfrac{z^4}{4!}$ in part 1. of Corollary 1.15 because $p(x) = x^2 + x$ satisfies $p(0) = 0$.

(ii) $T_4 f(x) = x + \dfrac{x^2}{2} + \dfrac{x^3}{3} + \dfrac{x^4}{4}$.

Hint: recall $\ln(1/a) = -\ln(a)$. Then use part 5. of Corollary 1.15 with $p(x) = -x$.

(iii) $T_4 f(x) = e - \dfrac{e}{2}x^2 + \dfrac{e}{6}x^4$.

Hint: this is the more difficult. Note $\cos(0) = 1$, so use the Taylor polynomial of e^z near $z = 1$ with $z = T_4 \cos x = 1 - \dfrac{x^2}{2} + \dfrac{x^4}{24}$.

(iv) $T_4 f(x) = 1 - x + \dfrac{x^3}{3} - \dfrac{x^4}{6}$.

Hint: look at (1.9).

5. Recall that $\lfloor n \rfloor$ denotes the largest integer smaller than n. This will be useful to stop the sums so that the powers of the variable do not exceed n. Unless indicated, use Corollary 1.15 with the right p (check, arrange that $p(0) = 0$ if needed).

(i) $T_n f(t) = \displaystyle\sum_{j=0}^{n} \frac{k^j}{j!} t^j$.

(ii) $T_n f(t) = \displaystyle\sum_{k=0}^{n} \frac{e}{k!} t^k$.

(iii) $T_n f(t) = \displaystyle\sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k}{k!} t^{2k}$.

(iv) $T_n f(t) = \displaystyle\sum_{k=0}^{\lfloor n/10 \rfloor} \frac{(-1)^k}{2k!} t^{10k}$.

(v) $T_n f(t) = 1 + 2\sum_{k=1}^{n} t^k$.
Hint: decompose f in partial fractions.

(vi) $T_n f(t) = \sum_{k=0}^{n} (-2)^k t^k$.

(vii) $T_n f(t) = \displaystyle\sum_{k=0}^{n} \frac{1}{2^{k+1}} t^k$.
Hint: to use part 2. of Corollary 1.15 you need to write $2 - t = 2(1 - t/2)$.

(viii) $T_n f(t) = \sum_{k=0}^{n} (T_k \exp(1))\, t^k$.

(ix) $T_n f(t) = \displaystyle\sum_{k=0}^{n} \frac{2k!}{(k!2^k)^2} t^k$.
Hint: for this problem, use the Binomial Theorem (see last year) or differentiate many times and find a 'general form' of the derivative of f at 0.

(x) $T_n f(t) = \displaystyle\sum_{k=0}^{n} \frac{1}{3}\left(1 + \frac{(-1)^k}{2^{k+1}}\right) t^k$.
Hint: decompose f in partial fractions.

(xi) $T_n f(t) = \displaystyle\sum_{k=0}^{\lfloor n/2 \rfloor} \frac{-1}{k} t^{2k}$.

(xii) $T_n f(t) = \displaystyle\sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \frac{(-1)^k 4^k}{2k+1!} t^{2k+1}$.
Hint: use a trigonometric formula.

6. (i) $T_2 f = 0$, $T_n f$ does not exist for $n \geq 3$.

(ii) $T_3^1 f = 1 + 3(x-1) + 3(x-1)^2 + (x-1)^3$.

(iii) $T_3^{-1} f = 1 - 3(x+1) + 3(x+1)^2 - (x+1)^3$.

7. (i) 7.

(ii) -1.

(iii) $-1/2$.

(iv) $-\infty$.

(v) $\alpha = 1$ and the limit is -2.

8. (i) a minimum.

(ii) a saddle point anyway (think about it). When $b \neq a + 1/6$, the leading term is quintic, when $b = a + 1/6$ and $a \neq -7/60$, the leading term is cubic, and, when $(a, b) = (-7/60, 1/20)$, the leading term is of order 7.

9. Use induction with l'Hôpital Rule.

10. Use the fact that if a polynomial p has a root at $x - a$, it is uniquely divisible by $x - a$, that is, there exists an unique polynomial r of degree $\deg(p) - 1$ such that $p(x) = r(x)(x - a)$.

1.5.2 Feedback for Sheet 1a

1. We use Theorem 1.5.

(i) We have $q(x) = 43+19(x-7)+\frac{11}{2}(x-7)^2$. Expanding and re-ordering the coefficients we get $q(x) = \dfrac{11}{2}x^2 - 58x + \dfrac{359}{2}$.

(ii) Note that we have three conditions for a polynomial with four coefficients, so one coefficient is free. Clearly, using Theorem 1.5, the first three terms of p are fixed, and we can choose whatever we like for the cubic term. Thus we have an infinite number of cubic polynomials

$$p(x) = 3 + 8(x-2) - \frac{1}{2}(x-2)^2 + c(x-2)^3,$$

for any non-zero constant $c \in \mathbb{R}$. For the first part we choose $c = 1$ and get

$$p(x) = x^3 - \frac{13}{2}x^2 + 22x - 23.$$

(iii) The final polynomial is clearly

$$p(x) = 4x^3 - 4x^2 - 3x + 1.$$

2. (i) To determine the largest domain of a rule f, you need to get the x's such that $f(x)$ make sense. In this case, $x + 25$ must be non negative, so $x + 25 \geq 0$, hence $D(f) = [-25, \infty)$.

(ii) From Theorem 1.5, p is the Taylor polynomial $T_3^0 f$ of f. Straightforward differentiation get you $p(0) = f(0) = 5$, $p'(0) = f'(0) = 1/10$, $p''(0) = f''(0) = -1/500$ and $p'''(0) = f'''(0) = 6/50000$. Hence

$$p(x) = 5 + \frac{x}{10} - \frac{x^2}{1000} + \frac{x^3}{5 \cdot 10^4}.$$

3. You are being asked to provide the Taylor polynomials of f up to order 2. Note that if you keep the Taylor polynomial is the shape (1.2), the Taylor polynomial of lower order, say $T_k^a f$, can be found from the Taylor polynomial of higher order, say $T_l^a f$, by cutting out all the terms $(x-a)^j$ of order $k < j \leq l$, from $T_l^a f$ (though, this must be at the **same a**). Hence we only calculate and give the answer as $T_2^a f(x)$ and in the shape of (1.2).

(i) $T_2 f(x) = 0$, $T_2^1 f(x) = 1+3(x-1)+3(x-1)^2$ and $T_2^2 f(x) = 8+12(x-2)+6(x-2)^2$.

(ii) $T_2^1 f(x) = 1 - (x-1) + (x-1)^2$ and $T_2^2 f(x) = \frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{8}(x-2)^2$.

(iii) $T_2^1 f(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2$.

(iv) $T_2^1 f(x) = (x-1) - \frac{1}{2}(x-1)^2$ and $T_2^{e^2} f(x) = 2 + e^{-2}(x-e^2) - \frac{e^{-4}}{2}(x-e^2)^2$.

(v) Let $g(x) = \ln(x)$. Because $\ln(\sqrt{x}) = \frac{1}{2}\ln(x)$, then $T_2^1 f = \frac{1}{2}T_2^1 g$ (think about it). Using the answer of (iv), $T_2^1 f(x) = \frac{1}{2}(x-1) - \frac{1}{4}(x-1)^2$.

(vi) $T_2 f(x) = 2x$ and $T_2^{\pi/4} f(x) = 1 - 2(x-\pi/4)^2$.

(vii) $T_2^{\pi} f(x) = -1 + \frac{1}{2}(x-\pi)^2$.

(viii) $T_2 f(x) = 1 - 2x + x^2$ and $T_2^1 f(x) = (x-1)^2$.

(ix) $T_2 f(x) = 1 - x + \dfrac{x^2}{2}$.

4. In the following you do not need to evaluate all powers of $(x-a)$, only the ones up to order 4 (because we want the Taylor polynomials of order 4).

(i) Use part 1. of Corollary 1.15 with $p(x) = x + x^2$. Because $p(0) = 0$, we can replace it into the fourth order Maclaurin polynomial of e^z (around $z = 0$), namely $T_4 e^z = 1 + z + \dfrac{z^2}{2!} + \dfrac{z^3}{3!} + \dfrac{z^4}{4!}$. We get

$$\begin{aligned} T_4 f(x) &= T_4\left(1 + (x+x^2) + \frac{(x+x^2)^2}{2!} + \frac{(x+x^2)^3}{3!} + \frac{(x+x^2)^4}{4!}\right) \\ &= 1 + x + \frac{3}{2}x^2 + \frac{7}{6}x^3 + \frac{25}{24}x^4. \end{aligned}$$

(ii) Recall that $\ln(1/a) = -\ln(a)$. Use part 5. of Corollary 1.15 with $p(x) = -x$. Then

$$T_4 f(x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4}.$$

(iii) Because $\cos(0) = 1$, we need to use the Taylor polynomial of e^z around $z = 1$, that is,

$$T_4^1 e^z = e + e(z-1) + e\frac{(z-1)^2}{2} + e\frac{(z-1)^3}{3} + e\frac{(z-1)^4}{4}.$$

Then we replace $z-1$ by the Maclaurin expansion of order 4 of

$$\cos x - 1 = -\frac{x^2}{2} + \frac{x^4}{24}.$$

Therefore,

$$T_4 f(x) = e - \frac{e}{2}x^2 + \frac{e}{6}x^4.$$

(iv) Use (1.9) in Theorem 1.11. We multiply together the Maclaurin polynomials of order 4 of e^{-x} and $\cos(x)$, namely,

$$\begin{aligned} T_4 e^{-x} &= 1 + (-x) + \frac{1}{2}(-x)^2 + \frac{1}{6}(-x)^3 + \frac{1}{24}(-x)^4, \\ T_4\cos(x) &= 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4. \end{aligned}$$

Hence, collecting all terms up to order 4,

$$T_4 f(x) = T_4\left(T_4 e^{-x} \cdot T_4\cos(x)\right) = 1 - x + \frac{x^3}{3} - \frac{x^4}{6}.$$

5. Recall that $\lfloor n \rfloor$ denotes the largest integer smaller than n. This will be useful to stop the sums so that the powers of the variable do not exceed n.

(i) Use part 1. of Corollary 1.15 with $p(t) = kt$. We get $T_n f(t) = \sum_{j=0}^{n} \frac{(kt)^j}{j!} = \sum_{j=0}^{n} \frac{k^j}{j!} t^j$.

(ii) Because $e^{1+t} = e \cdot e^t$, so $T_n f(t) = \sum_{k=0}^{n} \frac{e}{k!} t^k$.

(iii) Use part 1. of Corollary 1.15 with $p(t) = -t^2$. Because we need to stop at degree $n = 2k$, we only need to add the terms up to $k = \lfloor n/2 \rfloor$. Finally, we get $T_n f(t) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k}{k!} t^{2k}$.

(iv) Use part 4. of Corollary 1.15 with $p(t) = t^5$. Because we need to stop at degree $n = 10k$, we get $T_n f(t) = \sum_{k=0}^{\lfloor n/10 \rfloor} \frac{(-1)^k}{2k!} t^{10k}$.

(v) Decompose f in partial fractions: $f(t) = \frac{1+t}{1-t} = -1 + \frac{2}{1-t}$. Using the GP formula, you get $T_n f(t) = 1 + 2\sum_{k=1}^{n} t^k$.

(vi) Substitute $u = -2t$ in the GP formula for $1/(1-u)$. You get $T_n f(t) = \sum_{k=0}^{n} (-2)^k t^k$.

(vii) To use part 2. of Corollary 1.15, you need to write $2 - t = 2(1 - t/2)$. And so, $T_n f(t) = \frac{1}{2} \sum_{k=0}^{n} \left(\frac{t}{2}\right)^k = \sum_{k=0}^{n} \frac{1}{2^{k+1}} t^k$.

(viii) Let $g(t) = e^t$ and $h(t) = 1/(1-t)$. Then,

$$T_n g(t) = \sum_{k=0}^{n} \frac{t^k}{k!}, \quad T_n h(t) = \sum_{k=0}^{n} t^k.$$

Multiplying together we see that the term of degree k is

$$(1 + 1 + \tfrac{1}{2!} + \cdots + \tfrac{1}{k!}) t^k = (T_k \exp(1))\, t^k,$$

and so

$$T_n f(t) = \sum_{k=0}^{n} (T_k \exp(1))\, t^k.$$

(ix) Calculate the derivatives of $(1-t)^{-1/2}$, being careful with the minus signs (alternatively, use the Binomial Theorem). The term of order k in the Taylor polynomial is

$$\frac{1 \cdot 3 \cdots (2k-1)}{2 \cdot 4 \cdots 2k}.$$

To simplify it, note that the product of even terms

$$2 \cdot 4 \cdot \ldots \cdot 2n = 2^n (n!).$$

Hence, if you multiply top and bottom by the missing even terms, you get

$$T_n f(t) = \sum_{k=0}^{n} \frac{2k!}{(k! 2^k)^2} t^k.$$

(x) Decompose in partial fraction:

$$\frac{1}{(2-t-t^2)} = \frac{1}{3(2+t)} + \frac{1}{3(1-t)} = \frac{1}{6(1+(t/2))} + \frac{1}{3(1-t)}.$$

Using the Taylor polynomials of a geometric progression, we get

$$\begin{aligned} T_n f(t) &= \frac{1}{6}\left(\sum_{k=0}^{n}(-t/2)^k\right) + \frac{1}{3}\left(\sum_{k=0}^{n} t^k\right) \\ &= \sum_{k=0}^{n} \frac{1}{3}\left(1+\frac{(-1)^k}{2^{k+1}}\right) t^k. \end{aligned}$$

(xi) Using part 5. of Corollary 1.15 with $p(t) = -t^2$, we get

$$T_n f(t) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{-1}{k} t^{2k}.$$

(xii) Recall that $\sin t \cos t = \frac{1}{2}\sin(2t)$. Using part 3. of Corollary 1.15 with $p(t) = 2t$, we get

$$T_n f(t) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \frac{(-1)^k 4^k}{2k+1!} t^{2k+1}.$$

6. Recall that from Level 1, the rule $f(x)$ can be written as $f(x) = \begin{cases} x^3, & x \geq 0 \\ -x^3, & x < 0. \end{cases}$. It is clear that all the derivatives of f exist at $x = 1$ or $x = -1$. At $x = 0$, only $f(0) = f'(0) = f''(0) = 0$ exist, the others do not. And so, we have the following Taylor polynomials.

(i) $T_2 f = 0$, $T_n f$ does not exist for $n \geq 3$.

(ii) $T_3^1 f = 1 + 3(x-1) + 3(x-1)^2 + (x-1)^3$.

(iii) $T_3^{-1} f = 1 - 3(x+1) + 3(x+1)^2 - (x+1)^3$.

7. (i) At the limit, the fraction is of type $0/0$. To eliminate the ambiguity, it is quick to see that we need to go to order at least 3 (maybe higher, but let's start at 3). Recall that $T_3 \sin x = x - x^3/6$, $T_2 \cos x = 1 - x^2/2$ and so

$$T_3 \tan x = T_3\left(\frac{T_3 \sin x}{T_3 \cos x}\right) = \left(x - \frac{x^3}{6}\right)\left(1+\frac{x^2}{2}\right) = x + \frac{x^3}{3}.$$

We can now replace everything and get, up to third order,

$$\begin{aligned} T_3\left(x(1+\cos x) - 2\tan x\right) &= 2x - \frac{x^3}{2} - 2x - \frac{2x^3}{3} = -\frac{7}{3}x^3, \\ T_3\left(2x - \sin x - \tan x\right) &= 2x - x + \frac{x^3}{6} - x - \frac{x^3}{3} = -\frac{1}{6}x^3. \end{aligned}$$

And so, the quotient, hence the limit, is 7.

Note that we would need to differentiate 3 times to use l'Hôpital Rule (try it?).

(ii) We need to expand the functions around $x = 1$ to identify the leading order term. We get

$$T_n^1 \ln x = (x-1) - \frac{1}{2}(x-1)^2 + \dots.$$

And so, for the numerator, $T_n^1(1 - x + \ln x) = -\frac{1}{2}(x-1)^2 + \dots$. The denominator can also be written as

$$1 - \sqrt{2x - x^2} = 1 - \sqrt{1 - (x-1)^2}$$

which brings in nicely the expansion around $x = 1$. Recall that, around $z = 0$,

$$T_n\sqrt{1-z} = 1 - \frac{z}{2} + \dots,$$

and so our denominator has leading term

$$1 - \left(1 - \frac{(x-1)^2}{2}\right) = \frac{(x-1)^2}{2}.$$

Finally, we get the limit as the quotient of the two leading terms, here equal to -1. Note that l'Ĥopital Rule would have needed two derivatives. There is also another way to derive the leading order term of the denominator:

$$\begin{aligned} 1 - \sqrt{1-(x-1)^2} &= \frac{(1 - \sqrt{1-(x-1)^2})(1 + \sqrt{1-(x-1)^2})}{1 + \sqrt{1-(x-1)^2}} \\ &= \frac{1 - (1 - (x-1)^2)}{1 + \sqrt{1-(x-1)^2}} \approx \frac{(x-1)^2}{2} + \dots \end{aligned}$$

(iii) The two fractions tend to $+\infty$ as x tends to 0. It is easier to recast the expression as one fraction, namely $\frac{\ln(1+x) - x}{x\ln(1+x)}$ and evaluate the leading order terms. We have $T_n \ln(1+x) = x - \frac{x^2}{2} + \dots$, and so the elements of the fraction become

$$\begin{aligned} T_n\left(\ln(1+x) - x\right) &= -\frac{x^2}{2}, \\ T_n\left(x\ln(1+x)\right) &= \frac{x^2}{2}, \end{aligned}$$

and so the limit is $-1/2$.

(iv) We again have the difference '$\infty - \infty$'. Again, recast the expression as a fraction:

$$\frac{(\sin x)^2 - x}{x(\sin x)^2}.$$

The leading order terms are

$$\begin{aligned} T_n((\sin x)^2 - x) &= -x + \dots, \\ T_n(x(\sin x)^2) &= x^3 + \dots, \end{aligned}$$

and so the overall limit is $-\infty$.

(v) We start by establishing the leading order therm of the denominator around $x = 0$. Recall that

$$2T_n \sinh x = T_n(e^x - e^{-x}) = T_n e^x - T_n e^{-x} = 2x + \frac{2}{3}x^3 + \ldots,$$

and so $x - \sinh x = -\dfrac{x^3}{3} + \ldots$. Therefore we need to look at the numerator and fix α such that its leading term is at least cubic. The difficult term to deal with is

$$T_3\left(xe^{-\sin x}\right) = xT_3\left(1 - \sin x + \frac{1}{2}\sin^2 x\right) = x - x^2 + \frac{x^3}{2}.$$

And so, up to cubic order, the numerator is equal to

$$T_3\left(xe^{-\sin x} - \sin x + \alpha x^2\right) = x - x^2 + \frac{x^3}{2} - x + \frac{x^3}{6} + \alpha x^2 = (\alpha - 1)x^2 + \frac{2}{3}x^3.$$

Therefore we need to set $\alpha = 1$ and the limit is -2.

8. (i) We need to determine the leading order terms around $x = 0$ of

$$\begin{aligned} T_n \sin(x^2) &= x^2 - \frac{x^6}{6} + \ldots, \\ T_n(x^2 \cos x) &= x^2(1 - \frac{x^2}{2} + \ldots) = x^2 - \frac{x^4}{2}, \end{aligned}$$

and so the leading order terms of the function are

$$\frac{x^4}{2} - \frac{x^6}{6} + \ldots$$

and so $x = 0$ is a minimum.

(ii) The rule $f(x)$ is odd, $f(-x) = -f(x)$, and so the critical point $x = 0$ can only be a saddle point. We need to get the leading term of $f(x)$ at $x = 0$. We know that

$$T_n \sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{7!} = \ldots.$$

For the fraction, using that $(1 + z)^{-1} = 1 - z + z^2 - z^3 + \ldots$, we have

$$(x + ax^3)(1 + bx^2)^{-1} = (x + ax^3)(1 - bx^2 + b^2x^4 - b^3x^6 + \ldots),$$

and so the expansion of $f(x)$ is

$$(-1/6 + b - a)x^3 + (1/120 - b^2 + ab)x^5 + (-1/7! + b^3 - ab^2)x^7 + \ldots.$$

When $b \neq a + 1/6$, the leading term is cubic, when $b = a + 1/6$ and $a \neq -7/60$, the leading term is quintic, and, when $(a, b) = (-7/60, 1/20)$, the leading term is of order 7.

9. Consider $g(x) = f(x) - T_n^a f(x)$. Then $h_x(x) = \frac{g(x)}{(x-a)^n}$. Now, the $n-1$-th derivative of g and of $(x-a)^n$ are $f^{(n-1)}(x) - f^{(n-1)}(a) - f^{(n)}(x-a)$ and $(n!)(x-a)$, respectively. Clearly they are both 0 at $x = a$, so, using l'Hôpital Rule,

$$\lim_{x \to a} \frac{g^{(n-1)}(x)}{(n!)(x-a)} = \frac{f^{(n)}(a) - f^{(n)}(a)}{n!} = 0.$$

We can now re-iterate the process backward until we get

$$\lim_{x \to a} h_n(x) = \lim_{x \to a} \frac{g'(x)}{n(x-a)^{(n-1)}} = 0.$$

10. Set $a_0 = p(a)$. Let $q_1(x) = p(x) - a_0$, so $\deg(q) = \deg(p)$ and $q_1(a) = 0$. Hence, there exists a unique polynomial $r_1(x)$ of degree $\deg(q) - 1 = \deg(p) - 1$ such that $q_1(x) = r_1(x)(x-a)$. Therefore,

$$p(x) = a_0 + r_1(x)(x-a).$$

Now, let $a_1 = r_1(a)$ and let $q_2(x) = r_1(x) - a_1$. Again, $\deg(q_2) = \deg(r_1) = \deg(p) - 1$ and $q_2(a) = 0$. So, there exists an unique polynomial r_2 of degree $\deg(r_1) - 1 = \deg(p) - 2$ such that $q_2(x) = r_2(x)(x-a)$. Hence, $r_1(x) = a_1 + r_2(x)(x-a)$ and

$$p(x) = a_0 + (a_1 + r_2(x)(x-a))(x-a) = a_0 + a_1(x-a) + r_2(x)(x-a)^2.$$

The process can be re-iterated until the degree of r_n is 0 and the conclusion follows.

Chapter 2

Real Sequences

2.1 Lecture 7: Definitions, Limit of a Sequence

Reference: Stewart Chapter 12.1, Pages 710–720. (Edition?)

We will define **sequences** properly soon, but for the moment think of a sequence as just as an **infinitely long list of real numbers in a definite order**. For instance,

$$1, 2, 3, 4, \ldots \quad 1, 4, 9, 16, \ldots \quad -1, 1, -1, 1, \ldots \quad 1/2, 1/4, 1/8, \ldots \quad 1/4, 1/2, 1/16, 1/8, \ldots \tag{2.1}$$

are all sequences. We will be interested in the **limits of sequences**, i.e. whether they get closer and closer to a particular value as we take more and more terms. In the examples above, providing we make reasonable assumptions about how the **tail** of the sequence behaves, the first two sequences are tending to infinity, the third oscillates and so does not tend to a limit and the fourth and fifth tend to 0. Determining whether a sequence **converges** can be extremely difficult, so we will concentrate on a few situations and tools we can use. In particular, we will see how to link a sequence with a real function and use the connection with **limits of functions** to determine its convergence. You have seen and practiced limits of functions at Level 1. Further, more general information, will be the subject of Chapter 5.

Cautionary Tale 2.1. *The limit behaviour of a sequence is really determined by its* **(infinite) tail**, *not by its first initial numbers. We shall come back later to this point.*

2.1.1 Definition of a Sequence

Usually we use a more concise notation to define a sequence, by specifying a **general term**. A general sequence

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

will be denoted by $\{a_n\}_{n=1}^{\infty}$. The second sequence (2.1), i.e. $1, 4, 9, 16, \ldots$ corresponding to the function $f(n) = n^2$ can be denoted by $\{n^2\}_{n=1}^{\infty}$.

Definition 2.2. *A* **sequence (of real numbers)** *is the infinite ordered list of real numbers $f(1)$, $f(2)$, $f(3)$, ..., obtained from a function $f : \mathbb{N} \to \mathbb{R}$.*

Conversely, given a general sequence of real numbers $a_1, a_2, a_3, a_4, \ldots$, we can define a function $f : \mathbb{N} \to \mathbb{R}$ by setting $f(n) = a_n$. So, **a sequence of real numbers and a function $f : \mathbb{N} \to \mathbb{R}$ are the same thing**.

Cautionary Tale 2.3. *In principle,* **the function f defining a sequence has no need to have a formula $f(n)$** *for its rule. For instance, a sequence starting by $-\pi, e^2, 8, -4.67, \ldots$ is unlikely to have a 'reasonable rule', although it will still define a function.*

But, in the following examples, we have such rules.

Examples.

1. The first example of (2.1) is obtained from $f : \mathbb{N} \to \mathbb{R}$ given by $f(n) = n$.

2. The function $f : \mathbb{N} \to \mathbb{R}$, $f(n) = \begin{cases} 27, & \text{if } n \text{ is odd,} \\ n^2, & \text{if } n \text{ is even,} \end{cases}$
determines the sequence $27, 4, 27, 16, 27, 36, \ldots$.

3. Revisiting the other examples of (2.1), we get for $f : \mathbb{N} \to \mathbb{R}$

$$\text{(i) } f(n) = n^2, \text{ (ii) } f(n) = (-1)^n, \text{ (iii) } f(n) = 2^{-n}, \text{ (iv) } f(n) = \begin{cases} 2^{n+1}, & \text{if } n \text{ is odd,} \\ 2^{n-1}, & \text{if } n \text{ is even.} \end{cases}$$

Remark 2.4. *Some authors use the notation $(n^2)_{n=1}^{\infty}$ to emphasise that the* **order of the terms** *in the sequence matters (Curly brackets are used with sets where the order of the terms is irrelevant). Because the type with curly brackets is more common,* **we make this choice***.*

Our definition requires sequences to start with terms from $n = 1$. However, it can be sometimes convenient to start with a different value of n.

Examples. For instance the sequence $2, 3, \ldots$ could be denoted by $\{n\}_{n=2}^{\infty}$ or $\{n+1\}_{n=1}^{\infty}$. The sequence $\{n^2+n\}_{n=1}^{\infty}$ is $2, 6, 12 \ldots$. It can also be written as $\{n^2-n\}_{n=2}^{\infty}$. If we want to ignore the first few terms we could write $\{n^2+n\}_{n=4}^{\infty}$ or, with another sequence, $\left\{\dfrac{\ln(x-4)}{n^2}\right\}_{n=5}^{\infty}$.
All of that has **no bearing on limits**.

2.1.2 Limit of a Sequence

We are often interested in what happens to a sequence as n becomes large, in particular, whether it tends to a limit as $n \to \infty$. In Chapter 5 we shall define **limit** for any sequence, but it is quite complicated. Because you already know from Level 1 the notion of limit for functions, we shall use it now for an easier definition, BUT **working only for a reduced class of sequences** where we can calculate associated limits of functions.

Definition 2.5. *Let $\{a_n\}_{n=1}^{\infty}$ be obtained from a function $f : [1, \infty) \subseteq \mathbb{R} \to \mathbb{R}$, $a_n = f(n)$ whenever $n \in \mathbb{N}$,* **AND suppose that $\lim_{x\to\infty} f(x) = l$**, *$l \in \mathbb{R}$. Then the sequence is called* **convergent** *with* **limit $\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x) = l$**. *If one cannot find a limit, that is no such function f exists, the sequence is called* **divergent***.*

Examples.

1. Let $a_n = \dfrac{1}{n^2}$. To show that $\lim_{n\to\infty} a_n = 0$, consider $f : [1,\infty) \to \mathbb{R}$, $f(x) = \dfrac{1}{x^2}$. Now $a_n = f(n)$ and $\lim_{x\to\infty} \dfrac{1}{x^2} = 0$. Then, by definition, $\lim_{n\to\infty} a_n = 0$.

2. To determine whether the sequence $\{\dfrac{n^2}{3n^2+2n+1}\}_{n=1}^{\infty}$ is convergent, set $a_n = \dfrac{n^2}{3n^2+2n+1}$ and $f : [1,\infty) \to \mathbb{R}$, $f(x) = \dfrac{x^2}{3x^2+2x+1}$ so that $f(n) = a_n$ when $n \in \mathbb{N}$. Because

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^2}{3x^2+2x+1} = \lim_{x\to\infty} \frac{1}{3+2/x+1/x^2} = \frac{1}{3},$$

$\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x) = \dfrac{1}{3}$.

Definition 2.5 is typical of definitions in mathematics where we use an explicit second object (like the function f) to define a property of a first one (the limit of the sequence $\{a_n\}_{n=1}^{\infty}$). For such definitions to make sense we need to show that for **ANY choice** of the second object we arrive at the **same property**. In our case we need the following result.

Lemma 2.6. *Let $\{a_n\}_{n=1}^{\infty}$ be a* **convergent** *sequence with two functions $f, g : [1,\infty) \to \mathbb{R}$ such that $a_n = f(n) = g(n)$ and $\lim_{x\to\infty} f(x) = l_1$, $\lim_{x\to\infty} g(x) = l_2$, $l_1, l_2 \in \mathbb{R}$. Then $l_1 = l_2$.*

Proof. By contradiction, suppose $l_1 \neq l_2$. Take $\epsilon = |l_1 - l_2|/3 > 0$ so that the intervals $(l_i - \epsilon, l_1 + \epsilon)$ and $(l_2 - \epsilon, l_2 + \epsilon)$ do not intersect. By definition of the limit of functions as $x \to \infty$, there exists $m_1, m_2 > 0$ such that $|f(x) - l_1| < \epsilon$ when $x > m_1$ and $|g(x) - l_2| < \epsilon$ when $x > m_2$. Now, when $x > \max(m_1, m_2)$,

$$3\epsilon = |l_1 - l_2| \leq |l_1 - f(x)| + |f(x) - g(x)| + |l_2 - g(x)| < 2\epsilon + |f(x) - g(x)|.$$

And so, $\epsilon < |f(x) - g(x)|$ for all $x > \max(m_1, m_2)$. This is a contradiction because $f(n) = a_n = g(n)$ for all integer $n > \max(m_1, m_2)$. □

Cautionary Tale 2.7. *In the definition of limit it is important that f such that $f(n) = a_n$* **has a limit as $x \to \infty$**. *The following example shows why. The sequence $\{0\}_{n=1}^{\infty}$ is clearly convergent to 0. Indeed, using the definition, let $a_n = 0$ and $f : [1,\infty) \to \mathbb{R}$, $f(x) = 0$, then $\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x) = 0$.*

But, if we made another, 'strange', choice of function, things could go wrong. Choose $f_2 : \mathbb{R} \to \mathbb{R}$, $f_2(x) = \sin(2\pi x)$, then $f_2(n) = 0$ if $n \in \mathbb{N}$. However, $\lim_{x\to\infty} f_2(x)$ **does not exist**. *This is not a contradiction. One important hypothesis of the definition is that $\lim_{x\to\infty} f(x)$* **exists** *for (at least) one function f. That is the case for our first choice f, hence we can conclude, although it is not the case for our second choice f_2. Note that, because it has no limit, f_2* **tells us nothing about whether the sequence converges or not**.

2.1.3 Graphic Representations of Sequences

In the following Figures 2.1 and 2.2, we portray the sequences $\{1+\frac{(-1)^n}{n}\}_{n=1}^{\infty}$ and $\{1+\frac{\cos n}{n}\}_{n=1}^{\infty}$.

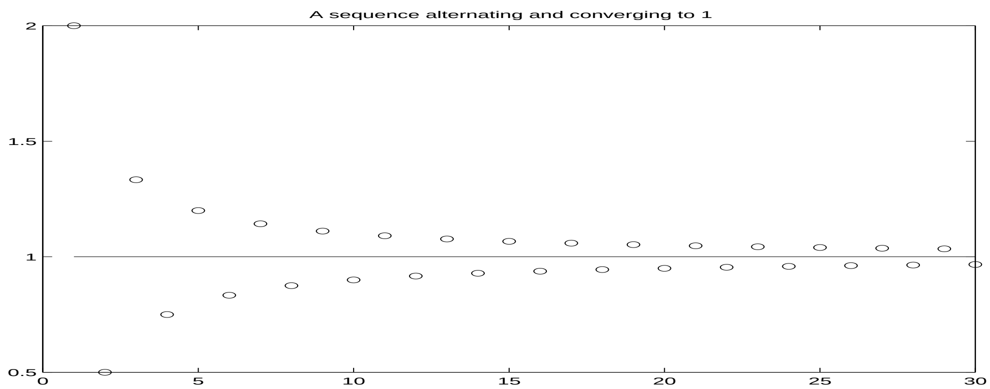


Figure 2.1: The first few values of a sequence alternating about the limit, like $1+\frac{(-1)^n}{n}$.

They both converge to 1. We can take $f_1(x) = 1 + \dfrac{\cos(\pi x)}{x}$ for the first sequence and $f_2(x) = 1 + \dfrac{\cos x}{x}$. Clearly,

$$1 = \lim_{x\to\infty}\left(1+\frac{\cos(\pi x)}{x}\right) = \lim_{x\to\infty}\left(1+\frac{\cos x}{x}\right).$$

The first one alternates up and down as it goes to 1, the second oscillates in no organised way towards 1.

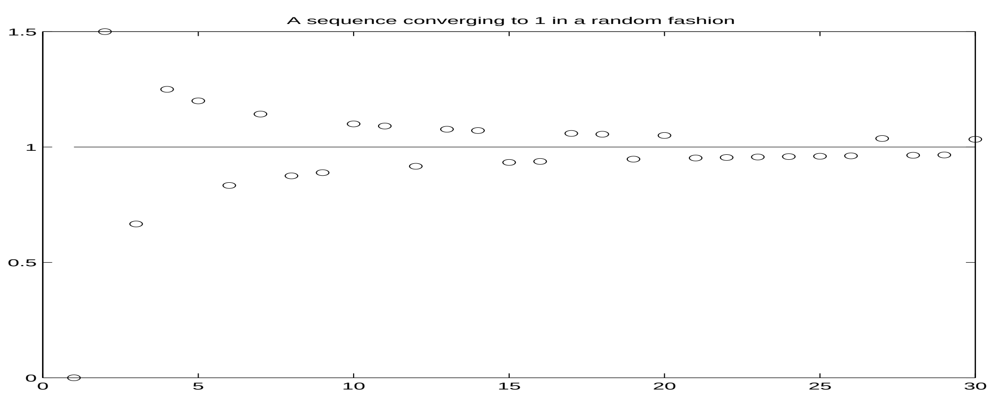


Figure 2.2: The first few values of a sequence converging in a random fashion to the limiting value, like $1+\frac{\cos n}{n}$.

2.2 Lecture 8: Algebra of Limits, Special Sequences

In the previous lecture the limit of a sequence was defined, linking the limit of a sequence to the limit of an associated function. We can use further this association and establish results that are very useful to evaluate explicitly limits of convergent sequences. In practice, using the **algebra of limits**, we can split limits into components we can evaluate, and put them together to find the final limit. In this lecture we also introduce the notion of infinite limit.

2.2.1 Infinite Limits

As with limit of functions, we can define what we mean by $\lim_{n\to\infty} a_n = +\infty$ or $-\infty$.

Definition 2.8. *Suppose that $f : [1, \infty) \subseteq \mathbb{R} \to \mathbb{R}$ and $f(n) = a_n$ whenever $n \in \mathbb{N}$. Furthermore, suppose that $\lim_{x\to\infty} f(x) = \pm\infty$, respectively. Then, $\lim_{n\to\infty} a_n = \pm\infty$, respectively.*

Example. The sequence $\{2^n\}_{n=1}^{\infty}$ tends to (positive) infinity because $\lim_{x\to\infty} 2^x = +\infty$ (as you have seen in Level 1).

Cautionary Tale 2.9. *Recall that $\pm\infty$ are* **not real numbers***. In particular, sequences with limits $+\infty$ or $-\infty$ are* **divergent***.*

Remark 2.10. *A divergent sequence does not need* **to grow unbounded***. It can oscillate between two values, for instance $\{(-1)^n\}_{n=1}^{\infty}$ (see Exercise 2(k) in Sheet 2a). Therefore a sequence* **either converges or diverges, but not both***.*

2.2.2 Algebra of Limits

We now have two theorems on limits of sequences which are very similar to theorems on limits of functions which you have seen earlier at Level 1. The first theorem is the algebra of limits for sequences and will be used to calculate limits of the type

$$\lim_{n\to\infty} \left(2 + \frac{\cos(n^2)}{3n} + n\ln(1 - 2/n)\right), \tag{2.2}$$

or

$$\lim_{n\to\infty} \cos\left(n \sin(1/n) - 1\right). \tag{2.3}$$

Theorem 2.11 (Algebra of Limits). *Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be* **convergent** *sequences and c be a real constant. Then:*

1. $\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$;
2. $\lim_{n\to\infty} (ca_n) = c\left(\lim_{n\to\infty} a_n\right)$;
3. $\lim_{n\to\infty} (a_nb_n) = \left(\lim_{n\to\infty} a_n\right) \cdot \left(\lim_{n\to\infty} b_n\right)$;

4. $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$, *providing* $b_n \neq 0$ *for any natural* n *and* $\lim_{n\to\infty} b_n \neq 0$.
5. *Let* f *be a continuous function around* $a = \lim_{n\to\infty} a_n$, *then the sequence* $\{f(a_n)\}_{n=1}^{\infty}$ *converges to* $f(a)$.

Proof. Those results are left as Exercises 7 and 8 in Sheet 2a. They follow directly from the definition of a limit and the same results for limits of functions you have seen at Level 1. □

This result is not enough to get the previous limit (2.2). We need one more result. The second theorem is the Sandwich Rule for sequences.

Theorem 2.12 (Squeeze Theorem, Sandwich Rule)**.** *Let* $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ *and* $\{c_n\}_{n=1}^{\infty}$ *be sequences such that* $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = l$ *and for all* n, $a_n \leq b_n \leq c_n$. *Then* $\lim_{n\to\infty} b_n = l$.

Proof. This result is left as Exercise 9 in Sheet 2a. It follows from the definition of a limit and the Squeeze Theorem for functions you have seen at Level 1. □

Cautionary Tale 2.13. *The results of Theorem 2.11 do not hold if the limits are not* **finite***. For instance, for the product of limits, consider* $a_n = n^2$, $b_n = 1/n$ *and* $c_n = 1/n^3$. *Then,*

$$\lim_{n\to\infty}(a_n b_n) = +\infty \neq 0 = \lim_{n\to\infty}(a_n c_n),$$

even if both $\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = 0$.

Examples. We can now revisit the previous examples to see how we can use Theorems 2.11 and 2.12.

1. For (2.2), we know that
$$\lim_{n\to\infty}\frac{-|\cos(n^2)|}{3n} \leq \lim_{n\to\infty}\frac{\cos(n^2)}{3n} \leq \lim_{n\to\infty}\frac{|\cos(n^2)|}{3n}.$$
We know that $\lim_{n\to\infty}\frac{|\cos(n^2)|}{3n} \leq \lim_{n\to\infty}\frac{1}{3n} = 0$, and so the limit of this fraction is 0. For $\lim_{n\to\infty} n\ln\left(1-\frac{2}{n}\right)$, we can use a Taylor polynomial expansion
$$\lim_{n\to\infty} n\ln\left(1-\frac{2}{n}\right) = \lim_{x\to\infty} x\ln\left(1-\frac{2}{x}\right) = \lim_{x\to\infty} x\cdot\frac{-2}{x} = -2.$$
And so we can collect everything together and we find the final limit as
$$\lim_{n\to\infty}\left(2+\frac{\cos(n^2)}{3n}+n\ln(1-2/n)\right) = 2+0-2 = 0.$$
2. For the second limit, using a Taylor polynomial expansion,
$$\lim_{n\to\infty} n\sin(1/n) = \lim_{x\to\infty} x\cdot(1/x) = 1.$$
And so,
$$\begin{aligned}\lim_{n\to\infty}\cos(n\sin(1/n)-1) &= \lim_{x\to\infty}\cos(x\sin(1/x)-1)\\ &= \cos\left(\lim_{x\to\infty} x\sin(1/x)-1\right) = \cos(0) = 1.\end{aligned}$$

2.2.3 Some Standard Convergent Sequences

We state next a few standard sequences with 'well known' limits. We can use the previous theorems to get from their behaviour the limits of more complicated sequences. The proofs are not always trivial!

Proposition 2.14. ***1.*** *Let* $p > 0$, $\lim_{n\to\infty} 1/n^p = 0$.

2. *The limit* $\lim_{n\to\infty} r^n = 0$ *when* $|r| < 1$. *It diverges when* $|r| > 1$.

3. *The limit* $\lim_{n\to\infty} n^{1/n} = 1$.

4. *If* $c > 0$ *then* $\lim_{n\to\infty} c^{1/n} = 1$.

5. *This is a very important limit:*

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e,$$

and has an equally important generalisation for $a \in \mathbb{R}$*:*

$$\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n = e^a.$$

6. *For all* $a, p > 0$, $\displaystyle\lim_{n\to\infty}\frac{n^p}{e^{an}} = 0$.

7. *For all* $p > 0$, $\displaystyle\lim_{n\to\infty}\frac{\ln n}{n^p} = 0$.

Proof. The proofs will mainly follow from the application of l'Hospital's Rule or the following idea. If f is a positive function such that $\lim_{x\to\infty}\ln(f(x))$ exists, because $\exp x$ and $\ln x$ are continuous on their domain of definition, the following limit makes sense

$$\lim_{x\to\infty} f(x) = \exp\left(\lim_{x\to\infty}\ln f(x)\right). \tag{2.4}$$

It is also straightforward to check that when $\lim_{x\to\infty}\ln(f(x)) = -\infty$,

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\exp\left(\ln f(x)\right) = 0. \tag{2.5}$$

1. Using (2.4),

$$\lim_{n\to\infty} n^{-p} = \exp\left(\lim_{x\to\infty}\ln x^{-p}\right) = \exp\left(-p\lim_{x\to\infty}\ln x\right) = 0.$$

2. Again, we shall be using (2.4) and the Squeeze Theorem. When $r > 0$,

$$\lim_{x\to\infty} x\ln r = \begin{cases} -\infty, & r < 1; \\ \infty, & r > 1. \end{cases}$$

And so, when $0 \le r < 1$,

$$\lim_{n\to\infty} r^n = \exp\left(\lim_{x\to\infty} x\ln r\right) = 0$$

and, when $r > 1$,

$$\lim_{n\to\infty} r^n = \exp\left(\lim_{x\to\infty} x\ln r\right) = \infty.$$

Moreover, when $-1 < r < 0$, $-|r| \le r \le 0$ and so, from the Squeeze Theorem, $\lim_{n\to\infty} r^n = 0$ and, obviously, r^n diverges when $r < -1$.

3. Using (2.4),

$$\lim_{x\to\infty} x^{1/x} = \exp\left(\lim_{x\to\infty} \ln x^{1/x}\right) = \exp\left(\lim_{x\to\infty} \frac{\ln x}{x}\right) = \exp(0) = 1,$$

having used l'Hospital Rule for the limit $\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1}{x} = 0$.

4. Using (2.4),

$$\lim_{x\to\infty} c^{1/x} = \exp\left(\lim_{x\to\infty} \ln c^{1/x}\right) = \exp\left(\lim_{x\to\infty} \frac{\ln c}{x}\right) = \exp(0) = 1.$$

5. Using (2.4), and Taylor series expansion for the last limit, we have for all $a \in \mathbb{R}$,

$$\lim_{n\to\infty} \left(1 + \frac{a}{n}\right)^n = \exp\left(\lim_{x\to\infty} x \ln(1 + \frac{a}{x})\right) = \exp\left(\lim_{x\to\infty} x \cdot \frac{a}{x}\right) = e^a.$$

6. Use the Squeeze Theorem because $0 \leq n^p \leq n^N$ for any integer $N > p$. Then, use N times l'Hospital Rule of the type '∞/∞':

$$\lim_{x\to\infty} \frac{x^N}{e^{ax}} = \lim_{x\to\infty} \frac{N!}{a^N e^{ax}} = 0.$$

7. Use l'Hospital Rule of the type '∞/∞' once. We get

$$\lim_{x\to\infty} \frac{\ln x}{x^p} = \lim_{x\to\infty} \frac{1/x}{px^{p-1}} = \lim_{x\to\infty} \frac{1}{px^p} = 0.$$

□

2.3 Lecture 9: Bounded and Monotone Sequences

We defined the convergence of a sequence and found some ways of evaluating limits. It may often be difficult to calculate the explicit limit of a sequence, so we split the problem into two stages:

1. show that a limit exists, then

2. calculate the limit (when it exists), even using numerical means.

It is only the first part that interests us here. We are going to determine sufficient conditions for the convergence of sequences.

2.3.1 Bounded Sequences

Convergence implies that only a finite number of terms of the sequence lie outside of a neighbourhood of this limit. Hence, the sequence is **bounded**, with the following formal definitions.

Definition 2.15. *A sequence $\{a_n\}_{n=1}^{\infty}$ is* **bounded below** *(***bounded above***) if there is $m \in \mathbb{R}$ ($M \in \mathbb{R}$) such that $a_n \geq m$ ($a_n \leq M$) for all $n \geq 1$. We say that a sequence is* **bounded** *if it is bounded below AND above, that is, if $m \leq a_n \leq M$ for all $n \geq 1$.*

Examples. The sequence $\{e^n + \cos n\}$ satisfies $0 \leq e - 1 \leq e^n + \cos n$ and so is bounded below ($m = e - 1$ or 0). Also, $-\pi/2 \leq \arctan n \leq \pi/2$ and so $\{\arctan n\}$ is bounded.

The sequence $\{(-1)^n\}_{n=1}^{\infty}$ is bounded BUT **not** convergent. The converse is however true which we next prove.

Theorem 2.16 (Convergence implies bounded)**.** *A convergent sequence is bounded.*

Proof. Let $\{a_n\}_{n=1}^{\infty}$ be a convergent sequence with limit a, induced by $f\colon [1, \infty) \to \mathbb{R}$ such that $a_n = f(n)$ and $\lim_{x\to\infty} f(x) = a$. Take $\epsilon = 1$. There exists an integer N such that $|f(x) - a| < \epsilon = 1$ for all $x \geq N$. Define $m = \min\{\ f(1), f(2), \ldots, f(N), a - 1\ \}$ and $M = \max\{\ f(1), f(2), \ldots, f(N), a + 1\ \}$, then $m \leq f(n) \leq M$ for all $n \in \mathbb{N}$. □

From the **contrapositive statement** (see Level 1) of Theorem 2.16, it follows that 'not bounded' implies 'not convergent': **an unbounded sequence is divergent**.

2.3.2 Convergent Sequences and Closed Bounded Intervals

Theorem 2.17 (Non-negative sequences)**.** *If $\{x_n\}_{n=1}^{\infty}$ is a convergent sequence of real numbers and if $x_n \geqslant 0$ for all $n \in \mathbb{N}$ then*

$$l = \lim_{n\to\infty} x_n \geqslant 0.$$

Similarly, if $x_n \leq 0$, then $l \leq 0$.

Proof. We will defer this until a later lecture when we meet the ϵ — N definition of convergence. □

What follows is an important property characterising the limit of a convergent sequence remaining in a closed bounded interval.

Corollary 2.18 (Closed bounded intervals contain their limits)**.** *If $\{x_n\}_{n=1}^{\infty}$ is a convergent sequence in the closed bounded interval $[a, b]$ then the limit $l \in [a, b]$.*

Proof. We show that $l \geq a$. A similar argument shows that $l \leq b$. Being a closed interval, it means that $l \in [a, b]$. Consider the sequence $\{y_n\}_{n=1}^{\infty}$ where $y_n = x_n - a$. Then $y_n \geq 0$ and so $\lim_{n\to\infty} y_n = l - a \geq 0$. □

2.4 Lecture 10: Monotone Sequences

As with functions, we can use the information that a sequence is increasing or decreasing.

Definition 2.19. *A sequence $\{a_n\}_{n=1}^{\infty}$ is* **increasing (or decreasing)** *if $a_n \leq a_{n+1}$ (or $a_n \geq a_{n+1}$) for all $n \geq 1$ or, in other words, if $a_1 \leq a_2 \leq a_3 \leq \ldots$ (increasing) or if $a_1 \geq a_2 \geq a_3 \geq \ldots$ (decreasing). A sequence that is either increasing or decreasing is called* **monotone***.*

Remark 2.20. *From our definitions, a sequence can be both increasing* **and** *decreasing. In that case it must be of constant value, say c. Formally such a sequence $\{c\}_{n=1}^{\infty}$ is monotone.*

Examples.

1. To determine whether the sequence $\left\{\dfrac{n}{n+1}\right\}$ is increasing or decreasing, we could simply calculate the sign of the difference $a_{n+1} - a_n$ where $a_n = \dfrac{n}{n+1}$. We get

$$a_{n+1} - a_n = \frac{n+1}{n+2} - \frac{n}{n+1} = \frac{(n+1)^2 - n(n+2)}{(n+1)(n+2)} = \frac{1}{(n+1)(n+2)} > 0.$$

   Hence $a_{n+1} > a_n$ and so $a_n \leq a_{n+1}$ and the sequence is increasing.

2. For the sequence $\left\{\dfrac{n}{n^2+1}\right\}$, let $a_n = \dfrac{n}{n^2+1}$. We consider

$$\begin{aligned} a_{n+1} - a_n &= \frac{n+1}{(n+1)^2+1} - \frac{n}{n^2+1} = \frac{(n+1)(n^2+1) - n((n+1)^2+1)}{(n^2+1)((n+1)^2+1)} \\ &= \frac{(n^3+n^2+n+1) - (n^3+2n^2+n+1)}{(n^2+1)((n+1)^2+1)} = -\frac{n^2}{(n^2+1)((n+1)^2+1)} < 0. \end{aligned}$$

   Hence $a_{n+1} < a_n$ and so $a_n \geq a_{n+1}$ and the sequence is decreasing.

When a sequence is obtained from a continuously differentiable function $f : [1, \infty) \to \mathbb{R}$, we can use Level 1 results about the monotonicity of functions to get the following result.

Proposition 2.21. *Let $f : [1, \infty) \subseteq \mathbb{R} \to \mathbb{R}$ be continuously differentiable and $\{a_n\}_{n=1}^{\infty}$ be the sequence obtained from f such that $a_n = f(n)$, $n \in \mathbb{N}$. If $f'(x) \geq 0$, $x \in [1, \infty)$, then $\{a_n\}_{n=1}^{\infty}$ is increasing, if $f'(x) \leq 0$, $x \in [1, \infty)$, then $\{a_n\}_{n=1}^{\infty}$ is decreasing.*

Proof. Observe that, from the Mean Value Theorem, and with $(n + 1) - n = 1$,

$$x_{n+1} - x_n = \frac{f(n+1) - f(n)}{(n+1) - n} = f'(\xi_n)$$

where $\xi_n \in (n, n + 1)$. If the derivative of f is non-negative then the sequence is increasing, if it is non-positive, the sequence is decreasing. ☐

We can revisit the two previous examples.

Examples.

1. Let $f(x) = \dfrac{x}{x+1}$. Then $f'(x) = \left(1 - \dfrac{1}{x+1}\right)' = \dfrac{1}{(x+1)^2} \geq 0$ for $x \geq 1$. and so, as expected, the sequence $\{f(n)\}_{n=1}^{\infty}$ is increasing.

2. Let $f(x) = \dfrac{x}{x^2+1}$. Then $f'(x) = \dfrac{1 - x^2}{(x^2+1)^2} \leq 0$ for $x \geq 1$. And so, the sequence $\{f(n)\}_{n=1}^{\infty}$ is decreasing.

Remark 2.22. *As a comment here, a minority of texts define 'increasing' as $x_1 < x_2 < \cdots$ and 'decreasing' as $x_1 > x_2 > \cdots$ which will be defined in Analysis later as* **strictly increasing** *and* **strictly decreasing***. Then, those texts use the terms 'non decreasing' for what we define here as* **increasing** *and 'non increasing' for* **decreasing***. This minor difference in terminology makes no difference to any of the results covered in this study block.*

2.4.1 Convergence of Monotone, Bounded Sequences

To be sure that a bounded sequence is convergent we need to add **one more condition**. The main result that we will consider about such sequences is as follows.

Theorem 2.23 (Monotone Convergence Theorem). *A monotone sequence converges if and only if it is bounded. More precisely, a bounded above and increasing sequence OR a bounded below and decreasing sequence are convergent.*

The proof of that theorem will have to wait until Analysis II (MA2731) in Term 2.

Remark 2.24. *We shall also see in Term 2 that the* **limit** *of a monotone convergent sequence $\{s_n\}_{n=1}^{\infty}$ can be determined by, respectively,*

$$\lim_{n\to\infty} x_n = \sup\{\, x_n : \ n \in \mathbb{N} \,\} \ (x_n \text{ increasing}), \qquad \lim_{n\to\infty} x_n = \inf\{\, x_n : \ n \in \mathbb{N} \,\} \ (x_n \text{ decreasing}).$$

We shall not give here precise definitions of the sup *and the* inf *of a set. This will happen in MA2731 next term. Simply take them as*

1. *the 'lowest upper bound' for the* sup*, like in*

$$\sup\{\ 1-1/n:\ n\in\mathbb{N}\}=\sup\{\ 0,1/2,2/3,3/4\cdots\}=1,$$

2. *the 'highest lower bound' for the* inf*, like in*

$$\inf\{\ 1/n:\ n\in\mathbb{N}\}=\inf\{\ 1,1/2,1/3,1/4\cdots\}=0.$$

This result indicates that we have a precise description of the limit of a monotone sequence, provided we know the definition and to calculate the sup and inf of a set. We illustrate what happens in Figures 2.3 and 2.4.

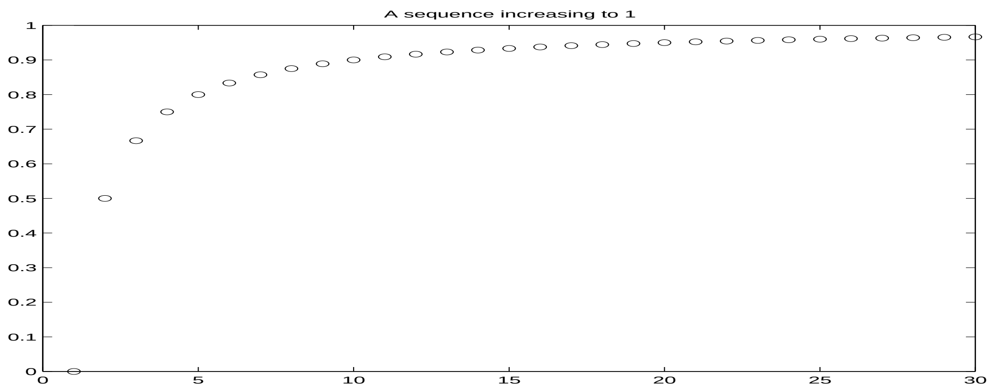


Figure 2.3: The first few values of a sequence increasing to 1, like $(1-\frac{1}{n})$.

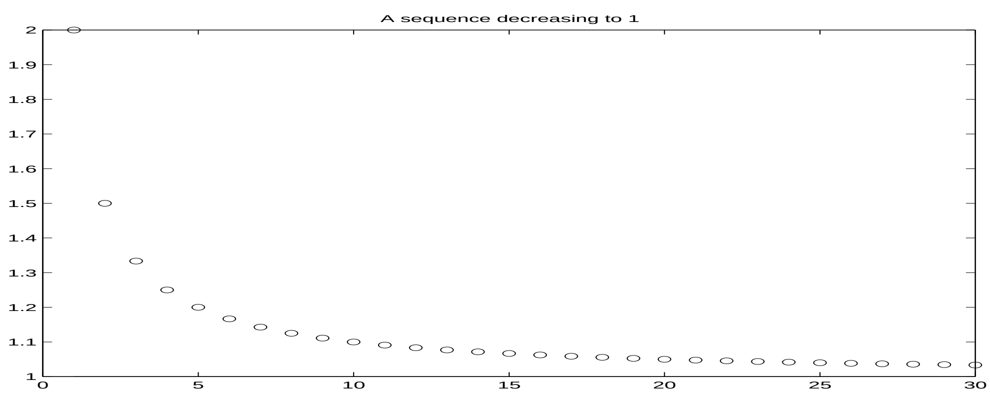


Figure 2.4: The first few values of a sequence decreasing to 1, like $(1+\frac{1}{n})$.

We now finish by looking at examples. We can calculate the limits using our previous techniques, but we show that they satisfy Theorem 2.23.

Examples. Consider the following sequences $\{x_n\}_{n=1}^{\infty}$.

1. The sequence $\{a_n\}_{n=1}^{\infty}$ where $a_n=\dfrac{1}{2n+3}$ is convergent because it is

(i) **bounded**, as clearly $0\le a_n$ and $a_n\le 1$ (since $2n+3>1$) for all $n>1$;

(ii) **monotone decreasing** because

$$a_{n+1}-a_n = \frac{1}{2(n+1)+3}-\frac{1}{2n+1} = \frac{(2n+1)-(2n+3)}{(2n+1)(2n+3)} = -\frac{2}{(2n+1)(2n+3)} < 0.$$

2. The sequence $\{\frac{\sqrt{n}}{n+1}\}_{n=1}^{\infty}$ is convergent because it is bounded by $0 \leq \frac{\sqrt{n}}{n+1} \leq 1$ and, defining $f : [1, \infty)$ by $f(x) = \frac{\sqrt{x}}{x+1}$, its derivative is negative when $x \geq 1$:

$$f'(x) = \frac{\frac{1}{2\sqrt{x}}(x+1) - \sqrt{x}}{(x+1)^2} = \frac{(x+1) - 2\sqrt{x}\sqrt{x}}{2\sqrt{x}(x+1)^2} = \frac{1-x}{2\sqrt{x}(x+1)^2} \leq 0.$$

And so the sequence is decreasing and bounded (hence convergent).

3. The sequence $\{\sqrt{n+1}-\sqrt{n}\}_{n=1}^{\infty}$ is the difference of two divergent sequences and thus the result about combining convergent sequences does not apply. However, it is decreasing because, either we could show that for $f(x) = \sqrt{x+1} - \sqrt{x}$, $f'(x) < 0$ for $x \geq 1$, or we could see that $f(x)$ simplifies (when multiplying the top and bottom by the conjugate quantity $\sqrt{n+1} + \sqrt{n}$) to $f(x) = \frac{1}{\sqrt{n+1} + \sqrt{n}}$. This function is clearly decreasing.

4. The sequence $\{\frac{2n^2 - 3n}{n+1}\}_{n=1}^{\infty}$ is an example of a divergent sequence. It is increasing because $f'(x) > 0$ when $x > 1$ for $f(x) = \frac{2x^2 - 3x}{x+1}$, but also unbounded because of the following estimate

$$\begin{aligned} x_n &= \frac{2n^2 - 3n}{n+1} = \frac{2(n+1)^2 - 7n - 2}{n+1} = 2(n+1) - \frac{7n+7-5}{n+1} \\ &= 2(n+1) - 7 + \frac{5}{n+1} \geq 2n - 5. \end{aligned}$$

Summary of Chapter 2

We have seen:

- the definition of a **sequence** and of a **convergent/divergent sequence**;
- how to calculate limits of many types of sequences;
- the meaning of **monotone (increasing/decreasing)** and **bounded sequences**, and that all bounded monotone sequences **converge**.
- To know the definition and properties of monotone bounded sequences.

2.5 Exercise Sheet 2

2.5.1 Exercise Sheet 2a

1. Fully revise the Level 1 material on limits of functions.

2. Determine which of the following sequences converge, and for those that do, find the limit.

(a) $\left\{\dfrac{n}{n+2}\right\}_{n=1}^{\infty}$; (b) $\left\{1-\dfrac{1}{2^n}\right\}_{n=1}^{\infty}$; (c) $\left\{\cos\left(\dfrac{1}{n}\right)\right\}_{n=1}^{\infty}$;
(d) $\left\{\dfrac{(-1)^n}{n^2}\right\}_{n=1}^{\infty}$; (e) $\left\{\ln\left(\dfrac{1}{n}\right)\right\}_{n=1}^{\infty}$; (f) $\left\{\sin\left(\dfrac{n\pi}{2}\right)\right\}_{n=1}^{\infty}$;
(g) $\left\{\dfrac{n}{2^n}\right\}_{n=1}^{\infty}$; (h) $\left\{\left(1-\dfrac{1}{n}\right)^n\right\}_{n=1}^{\infty}$; (i) $\left\{\dfrac{n}{2n+1}\right\}$;
(j) $\left\{\dfrac{\ln n}{n}\right\}$; (k) $\left\{(-1)^n\right\}$; (l) $\left\{\dfrac{(-1)^n}{n}\right\}$.

3. Discuss the convergence of the following sequences $\{x_n\}_{n=1}^{\infty}$ with the general terms:

(i) $x_n = (\sin n)(\sin(1/n))$,
(ii) $x_n = -n + \sqrt{n^2+3n}$,
(iii) $x_n = \dfrac{5n^3+3n+1}{15n^3+n^2+2}$,
(iv) $x_n = \dfrac{\sin(n^2+1)}{n^2+1}$,
(v) $x_n = \dfrac{\sqrt{n+2}-\sqrt{n+1}}{\sqrt{n+1}-\sqrt{n}}$,
(vi) $x_n = \dfrac{7n^4+n^2-2}{14n^4+5n-4}$,
(vii) $x_n = \dfrac{n^3+3n^2}{n+1} - n^2$,
(viii) $x_n = \left(\dfrac{n+1}{n}\right)^3 - n^3$,

and calculate their limit when it exists.

4. (i) Show that a polygon with n equal sides inscribed in a circle of radius r has perimeter $p_n = 2rn\sin\left(\dfrac{\pi}{n}\right)$.

(ii) By finding the limit of the sequence $\{p_n\}_{n=1}^{\infty}$, derive the formula for the circumference of a circle. (Hint: put $m = \pi/n$ and use l'Hôpital's rule.)

5. Show that $0 \le \dfrac{n!}{n^n} \le \dfrac{1}{n}$ and hence compute $\lim_{n\to\infty} \dfrac{n!}{n^n}$.

6. Calculate the limit (when it exists) of the sequences with the following general terms.

(i) $x_n = \dfrac{n^3+2n^2+1}{6n^3+n+4}$.

(ii) $x_n = \dfrac{n^2+n+1}{3n^2+4}$.

(iii) $x_n = \sqrt{n^4+n^2} - n^2$.

(iv) $x_n = \left(\dfrac{n+1}{n^2}\right)^4 - n^4$.

(v) $x_n = -n + \sqrt{n^2+n}$.

(vi) $x_n = \dfrac{\sin n}{n} + (\sqrt{n+1} - \sqrt{n})$.

(vii) $x_n = \dfrac{n^2+500n+1}{5n^2+3}$.

(viii) $x_n = \sqrt{n(n+1)} - \sqrt{n(n-1)}$.

7. Prove Statements 1-4 of Theorem 2.11.

8. Prove Statement 5 of Theorem 2.11.

9. Prove the Squeeze Theorem 2.12.

10. * We can prove a simplified version of the Contraction Mapping Theorem that some of you will see next term in MA2731. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a function such that $\phi'(x) \geqslant 0$ for all $x \in \mathbb{R}$. Given $x_0 \in \mathbb{R}$, define a sequence $\{x_n\}_{n=1}^{\infty}$ by $x_{n+1} = \phi(x_n)$, $n \geq 0$. Sequences of this type are often monotone. More precisely, show that the sequence is increasing if $\phi(x_0) > x_0$ and decreasing if $\phi(x_0) < x_0$.

Short Feedback for Sheet 2a

1. Bookwork.

2.

(a) 1;	(b) 1;	(c) 1;
(d) 0;	(e) divergent, $-\infty$;	(f) divergent;
(g) 0;	(h) $1/e$;	(i) $1/2$;
(j) 0;	(k) divergent;	(l) 0.

3. The limits are

(i) convergent.

(ii) $3/2$.

(iii) $1/3$.

(iv) 0.

(v) 1.

(vi) $1/2$.

(vii) ∞.

(viii) $-\infty$.

4. (i) Divide the polygon into triangles.

(ii) $2\pi r$.

5. 0.

6. The limits are

(i) 1/6.
(ii) 1/3.
(iii) 1/2.
(iv) $-\infty$.
(v) 1/2.
(vi) 0.
(vii) 1/5.
(viii) 1.

7. Use the equivalent results you have seen at Level 1 for the limits of functions.

8. Use the fact that f is continuous and that you have seen at Level 1 a similar result for limits of functions.

9. To use the Squeeze Theorem for limit of functions, you need to use a continuous function obtained by linking by straight lines two consecutive points of the sequence.

10. * Compare $x_1 - x_0$ in both cases.

2.5.2 Feedback for Sheet 2a

1. Bookwork.

2. See 'short feedback'.

3. (a) Let $x_n = (\sin n)(\sin(1/n))$. We know that $|\sin n| \leqslant 1$ and that $|\sin x| \leqslant |x|$ for all $x \in \mathbb{R}$, therefore

$$\lim_{n\to\infty} |x_n| = \lim_{n\to\infty} |(\sin n)(\sin(1/n))| \leqslant \lim_{n\to\infty} |\sin(1/n)| \leqslant \lim_{n\to\infty} 1/n = 0.$$

4. .

5. .

6. .

7. .

8. .

9. .

10. * Observe that by the Mean Value Theorem (a result you have met in Calculus at Level 1)

$$x_{n+1} - x_n = \phi(x_n) - \phi(x_{n-1}) = \phi'(\xi_n)(x_n - x_{n-1})$$

where ξ_n is some point between x_{n-1} and x_n. If the starting point x_0 is such that $x_1 = \phi(x_0) > x_0$ then this successively implies that $x_2 - x_1 \geqslant 0$, $x_3 - x_2 \geqslant 0$, $\cdots$, $x_{n+1} - x_n \geqslant 0$. That is $x_{n+1} \geqslant x_n \geqslant \cdots \geqslant x_2 \geqslant x_1 > x_0$, i.e. the sequence is increasing. Similar reasoning shows that if instead $x_1 < x_0$ then the sequence is decreasing.

Chapter 3

A flipped classroom approach to *Improper Integrals*

3.1 Self-study for Lecture 11: Improper Integrals — Type 1

Reference: Stewart Chapter 8.8, Pages 544–547.

In this part we will see how to integrate functions over infinite intervals. In the next part we will see that we can sometimes integrate functions even when they are not defined at one or more points covered by the range of integration.

Let

$$A(t) = \int_1^t \frac{1}{x^2}\,\mathrm{d}x = \left[-\frac{1}{x}\right]_1^t = 1 - \frac{1}{t}.$$

Thus however large we take t, we have $A(t) < 1$ so it is always finite. Also $\lim_{t\to\infty} A(t) = 1$, so it seems reasonable to say that

$$\int_1^\infty \frac{1}{x^2}\,\mathrm{d}x = 1.$$

However, we do need to be careful here. We cannot just deal with the ∞ as if it were a number, because it's not a number. There is another reason why we cannot just do this. Usually when we integrate, we use the Fundamental Theorem of Calculus to compute the integral as the difference between two evaluations of an antiderivative. For example, we might say $\int_a^b f(x)\,\mathrm{d}x = \Big[F(x)\Big]_a^b$ where F is an antiderivative for f. We have only proved the Fundamental Theorem of Calculus for finite intervals, so we can't just plug in infinity and hope for the best.

Definition 3.1 (Improper Integrals of Type 1)**.**

1. *If* $\int_a^t f(x)\,\mathrm{d}x$ *exists (i.e. can be computed and is finite) for all* $t \geq a$*, then we define*

$$\int_a^\infty f(x)\,\mathrm{d}x = \lim_{t\to\infty} \int_a^t f(x)\,\mathrm{d}x,$$

provided this limit exists.

2. *If* $\int_t^b f(x)\,\mathrm{d}x$ *exists (i.e. can be computed and is finite) for all* $t \leq b$*, then we define*

$$\int_{-\infty}^{b} f(x)\,\mathrm{d}x = \lim_{t\to-\infty} \int_t^b f(x)\,\mathrm{d}x,$$

provided this limit exists.

3. *If for some value of* a*,* $\int_a^\infty f(x)\,\mathrm{d}x$ *and* $\int_{-\infty}^a f(x)\,\mathrm{d}x$ *both exist then we define*

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \int_{-\infty}^{a} f(x)\,\mathrm{d}x + \int_a^\infty f(x)\,\mathrm{d}x.$$

If an improper integral exists we say that the integral is convergent. Otherwise we say that it is divergent.

Example. Determine whether $\int_1^\infty \frac{1}{x}\,\mathrm{d}x$ is convergent and if so evaluate it.

Solution. *We have*

$$\int_1^\infty \frac{1}{x}\,\mathrm{d}x = \lim_{t\to\infty} \int_1^t \frac{1}{x}\,\mathrm{d}x = \lim_{t\to\infty} \big[\ln x\big]_1^t = \lim_{t\to\infty} \ln t = \infty.$$

Since the limit does not exist, the integral is divergent.

In the lecture notes our next example was

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx.$$

Here though we give a harder example, you should complete this yourself — for practice.

Example. Determine $\int_{-\infty}^{\infty} \frac{e^{-x}}{1+e^{-2x}}\,\mathrm{d}x.$

Solution. *We have*

$$\int_{-\infty}^{\infty} \frac{e^{-x}}{1+e^{-2x}}\,\mathrm{d}x = \int_{-\infty}^{0} \frac{e^{-x}}{1+e^{-2x}}\,\mathrm{d}x + \int_0^\infty \frac{e^{-x}}{1+e^{-2x}}\,\mathrm{d}x,$$

provided both of these integrals are convergent.

We compute each of them separately. $\int_0^\infty \frac{e^{-x}}{1+e^{-2x}}\,\mathrm{d}x = \lim_{t\to\infty} \int_0^t \frac{e^{-x}}{1+e^{-2x}}\,\mathrm{d}x.$ *To compute this integral, we substitute* $u = e^{-x}$*. Then* $\frac{\mathrm{d}u}{\mathrm{d}x} = -e^{-x}$ *and so "*$-\,\mathrm{d}u = e^{-x}\,\mathrm{d}x$*". When* $x = 0$*,* $u = 1$ *and when* $x = t$*,* $u = e^{-t}$*.*

Consequently

$$\lim_{t\to\infty}\int_0^t \frac{e^{-x}}{1+e^{-2x}}\,\mathrm{d}x = \lim_{t\to\infty}\int_1^{e^{-t}} \frac{-1}{1+u^2}\,\mathrm{d}u = \lim_{t\to\infty}\left[-\tan^{-1}u\right]_1^{e^{-t}} = \lim_{t\to\infty}(-\tan^{-1}(e^{-t})+\tan^{-1}1).$$

Because $\tan$ *is continuous on* $(-\pi/2,\pi/2)$*, the function* $\tan^{-1}$ *is continuous and so*

$$\lim_{t\to\infty}\tan^{-1}(e^{-t}) = \tan^{-1}\left(\lim_{t\to\infty}e^{-t}\right) = \tan^{-1}(0) = 0.$$

Hence $\displaystyle\lim_{t\to\infty}\int_0^t \frac{e^{-x}}{1+e^{-2x}}\,\mathrm{d}x = \pi/4$.

On the other hand $\displaystyle\int_{-\infty}^0 \frac{e^{-x}}{1+e^{-2x}}\,\mathrm{d}x = \lim_{t\to-\infty}\int_t^0 \frac{e^{-x}}{1+e^{-2x}}\,\mathrm{d}x = \lim_{t\to-\infty}(-\tan^{-1}1+\tan^{-1}(e^{-t}))$.

Now as $t\to-\infty$, $e^{-t}\to\infty$ *and since* $\tan^{-1}$ *is continuous,* $\displaystyle\lim_{t\to-\infty}\tan^{-1}(e^{-t}) = \lim_{y\to\infty}\tan^{-1}y = \pi/2$.

Hence $\displaystyle\lim_{t\to-\infty}\int_t^0 \frac{e^{-x}}{1+e^{-2x}}\,\mathrm{d}x = \pi/2-\pi/4 = \pi/4$. *Consequently*

$$\int_{-\infty}^{\infty} \frac{e^{-x}}{1+e^{-2x}}\,\mathrm{d}x = \pi/4+\pi/4 = \pi/2.$$

Exercise 3.2. *Determine whether the following integrals are convergent and if so evaluate them.*

(a) $\displaystyle\int_1^\infty \frac{1}{x^3}\,\mathrm{d}x$; *(b)* $\displaystyle\int_{-\infty}^0 e^x\,\mathrm{d}x$; *(c)* $\displaystyle\int_{-\infty}^\infty \frac{1}{1+x^2}\,\mathrm{d}x$; *(d)* $\displaystyle\int_{-\infty}^\infty x^3\,\mathrm{d}x$.

Summary

We have seen:

- that when we integrate over an infinite range, we cannot just treat ∞ like a number;
- how to evaluate integrals over an infinite range using a limit.

3.2 Self-study for Lecture 11: Improper Integrals — Type 2

Reference: Stewart Chapter 8.4, Pages 547–550.

So far we have only integrated functions that are continuous or at least consist of a number of continuous pieces. This is because whenever we compute a definite integral, we are using the Fundamental Theorem of Calculus and this only applies to continuous functions. In this session we will see how we can integrate functions that are discontinuous. In some cases the functions may not even be defined at all points in the range of integration.

Suppose that f is continuous on the interval $[a, b)$ but has a vertical asymptote at $x = b$, that is, f is undefined at $x = b$. We can still try to find the area beneath the graph of f. Let

$$A(t) = \int_a^t f(x)\,\mathrm{d}x.$$

Suppose that $A(t)$ tends to a finite limit as $t \to b^-$, then it would be reasonable to define the area under f to be $\lim_{t\to b^-} A(t)$ and say that

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{t\to b^-} f(x)\,\mathrm{d}x.$$

Definition 3.3 (Improper Integrals of Type 2)**.**

1. *Suppose that f is continuous on $[a, b)$ and discontinuous at b. (f does not even have to be defined at $x = b$.) Then we define*

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{t\to b^-} \int_a^t f(x)\,\mathrm{d}x,$$

providing this limit exists.

2. *Suppose that f is continuous on $(a, b]$ and discontinuous at a. (f does not even have to be defined at $x = a$.) Then we define*

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{t\to a^+} \int_t^b f(x)\,\mathrm{d}x,$$

providing this limit exists.

3. *Suppose that f is continuous on $[a, b]$ except at $c \in (a, b)$ where it may not even be defined and that $\int_a^c f(x)\,\mathrm{d}x$ and $\int_c^b f(x)\,\mathrm{d}x$ both exist. Then we define*

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x.$$

Again we describe integrals as convergent or divergent depending on whether they exist.

Example. Determine whether $\int_{-3}^{1} \frac{x}{\sqrt{9-x^2}}\, \mathrm{d}x$ is convergent and if so evaluate it.

Solution. *We have*

$$\int_{-3}^{1} \frac{x}{\sqrt{9-x^2}}\, \mathrm{d}x = \lim_{t\to -3^+} \int_{t}^{1} \frac{x}{\sqrt{9-x^2}}\, \mathrm{d}x = \lim_{t\to -3^+} \left[-\sqrt{9-x^2}\right]_t^1 = \lim_{t\to -3^+} (\sqrt{9-t^2} - \sqrt{8}).$$

As $t \to -3^+$, $9 - t^2 \to 0^+$ *and so* $\sqrt{9-t^2} \to 0$. *Hence* $\int_{-3}^{1} \frac{x}{\sqrt{9-x^2}}\, \mathrm{d}x = -\sqrt{8}$.

Example. Determine whether $\int_{-1}^{3} \frac{1}{x^2}\, \mathrm{d}x$ is convergent and if so evaluate it.

Solution. *We have*

$$\int_{-1}^{3} \frac{1}{x^2}\, \mathrm{d}x = \int_{-1}^{0} \frac{1}{x^2}\, \mathrm{d}x + \int_{0}^{3} \frac{1}{x^2}\, \mathrm{d}x,$$

providing both of these integrals are convergent. We will try to evaluate the first one.

$$\int_{-1}^{0} \frac{1}{x^2}\, \mathrm{d}x = \lim_{t\to 0^-} \int_{-1}^{t} \frac{1}{x^2}\, \mathrm{d}x = \lim_{t\to 0^-} \left[-\frac{1}{x}\right]_{-1}^{t} = \lim_{t\to 0^-} \left(-\frac{1}{t} - 1\right).$$

This limit does not exist, so the original integral is divergent.

Here is a completely incorrect solution.

Bad Solution: Suppose we fail to notice that the function $\frac{1}{x^2}$ is not continuous at 0. If we just use the Fundamental Theorem of Calculus we get

$$\int_{-1}^{3} \frac{1}{x^2}\, \mathrm{d}x = \left[-\frac{1}{x}\right]_{-1}^{3} = -\frac{4}{3}.$$

This cannot possibly be correct because we have integrated a positive function and obtained a negative answer.

Exercise 3.4. *Find:*

(a) $\int_{1}^{2} \frac{1}{1-x}\, \mathrm{d}x$; *(b)* $\int_{0}^{1} \frac{1}{\sqrt{1-x}}\, \mathrm{d}x$; *(c)* $\int_{1}^{4} \frac{1}{(x-2)^{2/3}}\, \mathrm{d}x$; *(d)* $\int_{0}^{\infty} \frac{1}{\sqrt{x}}\, \mathrm{d}x$.

Summary

We have seen:

- that more care is required to integrate functions when they are not continuous;
- how to evaluate integrals when the integrand is not continuous / undefined at certain points using a limit.

3.3 Homework for improper integrals, Types 1 and 2

1. Finish off any remaining in-class exercises.
2. Determine which of the following integrals exist and evaluate them when possible.

   (a) $\displaystyle\int_1^\infty \frac{1}{(3x+1)^2}\,\mathrm{d}x$; (b) $\displaystyle\int_0^\infty \frac{1}{x^2+5x+6}\,\mathrm{d}x$; (c) $\displaystyle\int_0^\infty \cos x\,\mathrm{d}x$;

   (d) $\displaystyle\int_0^\infty e^{-x}\cos x\,\mathrm{d}x$; (e) $\displaystyle\int_0^2 x^2\ln x\,\mathrm{d}x$; (f) $\displaystyle\int_0^{\pi/2} \tan x\,\mathrm{d}x$;

   (g) $\displaystyle\int_0^1 \frac{e^{1/x}}{x^2}\,\mathrm{d}x$; (h) $\displaystyle\int_{-\infty}^\infty \frac{1}{x^2+4}\,\mathrm{d}x$; (i) $\displaystyle\int_0^\infty \frac{1}{\sqrt{x}(x+1)}\,\mathrm{d}x$;

   [Hint: It should be helpful to know that for any $k>0$, $\lim_{x\to 0^+} x^k\ln x = 0$. How do we prove this?]

3. Find the values of p for which the following integrals exist and evaluate the integrals when they exist.

   (a) $\displaystyle\int_0^1 x^p\,\mathrm{d}x$; (b) $\displaystyle\int_1^\infty x^p\,\mathrm{d}x$; (c) $\displaystyle\int_e^\infty \frac{1}{x(\ln x)^p}\,\mathrm{d}x$.

4. (a) Show that $e^{-x^2}\le e^{-x}$ for $x\ge 1$.

   (b) Determine $\displaystyle\int_1^\infty e^{-x}\,\mathrm{d}x$.

   (c) What can you deduce about $\displaystyle\int_1^\infty e^{-x^2}\,\mathrm{d}x$?

5. If f is continuous on $[0,\infty)$, then its Laplace transform is a function F for which the rule is $F(s)=\displaystyle\int_0^\infty f(t)e^{-st}\,\mathrm{d}t$, and the domain of F consists of those s for which the integral converges.

   Find the Laplace transform of:

   (a) $f(t)=1$; (b) $f(t)=e^t$; (c) $f(t)=t$.

6. The definitions of the trigonometric functions and hyperbolic functions can be extended to all complex numbers. Use Euler's theorem concerning e^{ix} and the definition of $\sinh$ and $\cosh$ to show that $\sinh x = -i\sin(ix)$ and $\cosh x=\cos ix$.
7. In this question we have a random mixture of integrals. You have to work out the method required. A couple of them are not too hard, most of them are slightly tricky and some are very tricky. Evaluate:

(a) $\displaystyle\int_0^1 \frac{1}{\sqrt{x^2+4}}\,\mathrm{d}x$; (b) $\displaystyle\int_0^1 \frac{e^x}{\sqrt{e^{2x}+1}}\,\mathrm{d}x$; (c) $\displaystyle\int_{-1}^2 \left|x-x^2\right|\,\mathrm{d}x$;

(d) $\displaystyle\int \frac{x^4}{x^{10}+16}\,\mathrm{d}x$; (e) $\displaystyle\int_0^{\pi/4} \cos^2 t\tan^2 t\,\mathrm{d}t$; (f) $\displaystyle\int_{\pi/4}^{\pi/3} \sin(4x)\cos(3x)\,\mathrm{d}x$;

(g) $\displaystyle\int \frac{1}{\sqrt{x^2+4x+8}}\,\mathrm{d}x$; (h) $\displaystyle\int \frac{1}{e^x-e^{-x}}\,\mathrm{d}x$; (i) $\displaystyle\int \frac{1}{\sqrt{x+1}-\sqrt{x}}\,\mathrm{d}x$;

(j) $\displaystyle\int_{-1}^1 \frac{e^{\tan^{-1}(y)}}{1+y^2}\,\mathrm{d}y$; (k) $\displaystyle\int x\sin^2 x\,\mathrm{d}x$; (l) $\displaystyle\int e^{x+e^x}\,\mathrm{d}x$;

(m) $\displaystyle\int \frac{1}{1+e^x}\,\mathrm{d}x$; (n) $\displaystyle\int \ln(x^2-1)\,\mathrm{d}x$; (o) $\displaystyle\int_1^3 r^4\ln r\,\mathrm{d}r$.

8. Challenge Question: Find the value of the constant C for which the integral

$$\int_0^\infty \left(\frac{x}{x^2+1}-\frac{C}{x+1}\right)\mathrm{d}x$$

exists. Evaluate the integral for this value of C.

3.4 Feedback

3.4.1 Feedback on in-class exercises

Lecture 1: Improper Integrals — Type 1

Feedback on In-Class Exercises

Exercise 1:

(a)

$$\int_1^\infty \frac{1}{x^3}\,\mathrm{d}x = \lim_{t\to\infty}\int_1^t \frac{1}{x^3}\,\mathrm{d}x = \lim_{t\to\infty}\left[-\frac{1}{2x^2}\right]_1^t = \lim_{t\to\infty}\left(-\frac{1}{2t^2}+\frac{1}{2}\right) = \frac{1}{2}.$$

(b)

$$\int_{-\infty}^0 e^x\,\mathrm{d}x = \lim_{t\to-\infty}\int_t^0 e^x\,\mathrm{d}x = \lim_{t\to-\infty}\left[e^x\right]_t^0 = \lim_{t\to-\infty}(1-e^t) = 1.$$

(c)

$$\int_{-\infty}^\infty \frac{1}{1+x^2}\,\mathrm{d}x = \int_{-\infty}^0 \frac{1}{1+x^2}\,\mathrm{d}x + \int_0^\infty \frac{1}{1+x^2}\,\mathrm{d}x,$$

providing both of these integrals converge. Now

$$\int_0^\infty \frac{1}{1+x^2}\,\mathrm{d}x = \lim_{t\to\infty}\int_0^t \frac{1}{1+x^2}\,\mathrm{d}x = \lim_{t\to\infty}\left[\tan^{-1}x\right]_0^t = \lim_{t\to\infty}\tan^{-1}t = \frac{\pi}{2}.$$

Similarly

$$\int_{-\infty}^{0} \frac{1}{1+x^2}\,\mathrm{d}x = \lim_{t\to-\infty} \int_t^0 \frac{1}{1+x^2}\,\mathrm{d}x = \lim_{t\to-\infty} \left[\tan^{-1} x\right]_t^0 = \lim_{t\to-\infty} (-\tan^{-1} t) = \frac{\pi}{2}.$$

Hence

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,\mathrm{d}x = \int_{-\infty}^{0} \frac{1}{1+x^2}\,\mathrm{d}x + \int_0^{\infty} \frac{1}{1+x^2}\,\mathrm{d}x = \pi/2 + \pi/2 = \pi.$$

(d)

$$\int_{-\infty}^{\infty} x^3\,\mathrm{d}x = \int_{-\infty}^{0} x^3\,\mathrm{d}x + \int_0^{\infty} x^3\,\mathrm{d}x,$$

providing both of these integrals converge. Now

$$\int_0^{\infty} x^3\,\mathrm{d}x = \lim_{t\to\infty} \int_0^t x^3\,\mathrm{d}x = \lim_{t\to\infty} \left[\frac{x^4}{4}\right]_0^t = \lim_{t\to\infty} \frac{t^4}{4} = \infty.$$

So at least one of the two integrals diverges and hence $\displaystyle\int_{-\infty}^{\infty} x^3\,\mathrm{d}x$ diverges.

Lecture 2: Improper Integrals — Type 2

Feedback on In-Class Exercises

Exercise 1:

(a) $\dfrac{1}{1-x}$ is undefined at $x = 1$ but is continuous on $(1, 2]$. Hence

$$\int_1^2 \frac{1}{1-x}\,\mathrm{d}x = \lim_{t\to 1^+} \int_t^2 \frac{1}{1-x}\,\mathrm{d}x = \lim_{t\to 1^+} \left[-\ln|1-x|\right]_t^2 = \lim_{t\to 1^+} \ln(t-1) = -\infty.$$

So the integral diverges.

(b) $\dfrac{1}{\sqrt{1-x}}$ is undefined at $x = 1$ but is continuous on $[0, 1)$. Hence

$$\int_0^1 \frac{1}{\sqrt{1-x}}\,\mathrm{d}x = \lim_{t\to 1^-} \int_0^t \frac{1}{\sqrt{1-x}}\,\mathrm{d}x = \lim_{t\to 1^-} \left[-2\sqrt{1-x}\right]_0^t = \lim_{t\to 1^-} (2 - 2\sqrt{1-t}) = 2.$$

(c) $\dfrac{1}{(x-2)^{2/3}}$ is undefined at $x = 2$ but is continuous on $[1, 4] \setminus \{2\}$. Hence

$$\int_1^4 \frac{1}{(x-2)^{2/3}}\,\mathrm{d}x = \int_1^2 \frac{1}{(x-2)^{2/3}}\,\mathrm{d}x + \int_2^4 \frac{1}{(x-2)^{2/3}}\,\mathrm{d}x,$$

providing both of these integrals converge. Now

$$\int_1^2 \frac{1}{(x-2)^{2/3}}\,\mathrm{d}x = \lim_{t\to 2^-}\int_1^t \frac{1}{(x-2)^{2/3}}\,\mathrm{d}x = \lim_{t\to 2^-}\left[3(x-2)^{1/3}\right]_1^t$$
$$= \lim_{t\to 2^-}(3(t-2)^{1/3}+3) = 3$$

and

$$\int_2^4 \frac{1}{(x-2)^{2/3}}\,\mathrm{d}x = \lim_{t\to 2^+}\int_t^4 \frac{1}{(x-2)^{2/3}}\,\mathrm{d}x = \lim_{t\to 2^+}\left[3(x-2)^{1/3}\right]_t^4$$
$$= \lim_{t\to 2^+}(3\times 2^{1/3} - 3(t-2)^{1/3}) = 3\times 2^{1/3}.$$

Hence

$$\int_1^4 \frac{1}{(x-2)^{2/3}}\,\mathrm{d}x = \int_1^2 \frac{1}{(x-2)^{2/3}}\,\mathrm{d}x + \int_2^4 \frac{1}{(x-2)^{2/3}}\,\mathrm{d}x = 3 + 3\times 2^{1/3}.$$

(d) This integral is improper for two reasons. First, the range of integration is infinite and second, $\frac{1}{\sqrt{x}}$ is undefined at 0. However $\frac{1}{\sqrt{x}}$ is continuous on $(0,\infty)$. So

$$\int_0^\infty \frac{1}{\sqrt{x}}\,\mathrm{d}x = \int_0^1 \frac{1}{\sqrt{x}}\,\mathrm{d}x + \int_1^\infty \frac{1}{\sqrt{x}}\,\mathrm{d}x$$

provided both of these integrals converge. We have

$$\int_1^\infty \frac{1}{\sqrt{x}}\,\mathrm{d}x = \lim_{t\to\infty}\int_1^t \frac{1}{\sqrt{x}}\,\mathrm{d}x = \lim_{t\to\infty}\left[2\sqrt{x}\right]_1^t = \lim_{t\to\infty} 2\sqrt{t} - 2 = \infty.$$

So, at least one of the two integrals diverges and hence $\int_0^\infty \frac{1}{\sqrt{x}}\,\mathrm{d}x$ does not converge.

3.4.2 Homework feedback for improper integrals

Homework solutions for Improper Integrals, Types 1 and 2

2. (a)

$$\int_1^\infty \frac{1}{(3x+1)^2}\,\mathrm{d}x = \lim_{t\to\infty}\int_1^t \frac{1}{(3x+1)^2}\,\mathrm{d}x = \lim_{t\to\infty}\left[-\frac{1}{3(3x+1)}\right]_1^t$$
$$= \lim_{t\to\infty}\left[-\frac{1}{3(3t+1)} + \frac{1}{12}\right] = \frac{1}{12}.$$

(b) We have

$$\int_0^\infty \frac{1}{x^2+5x+6}\,\mathrm{d}x = \lim_{t\to\infty}\int_0^t \frac{1}{x^2+5x+6}\,\mathrm{d}x.$$

We will determine $\int_0^t \frac{1}{x^2+5x+6}\,\mathrm{d}x$ using the method of partial fractions.
Factorising gives $x^2+5x+6=(x+2)(x+3)$ and so

$$\frac{1}{x^2+5x+6} = \frac{A}{x+2}+\frac{B}{x+3},$$

where A and B are constants to be determined.
Standard calculations yield $A=1$ and $B=-1$. So

$$\int_0^t \frac{1}{x^2+5x+6}\,\mathrm{d}x = \int_0^t \frac{1}{x+2}\,\mathrm{d}x - \int_0^t \frac{1}{x+3}\,\mathrm{d}x = \big[\ln|x+2|\big]_0^t - \big[\ln|x+3|\big]_0^t$$
$$= \ln\left(\frac{t+2}{t+3}\right)+\ln\left(\frac{3}{2}\right).$$

Hence

$$\lim_{t\to\infty}\int_0^t \frac{1}{x^2+5x+6}\,\mathrm{d}x = \lim_{t\to\infty}\ln\left(\frac{t+2}{t+3}\right)+\ln\left(\frac{3}{2}\right).$$

As $t\to\infty$, $\frac{t+2}{t+3}\to 1$ and since $\ln$ is continuous at 1, we can use the result from Continuity III which allows us to say that

$$\lim_{t\to\infty}\ln\left(\frac{t+2}{t+3}\right) = \ln\lim_{t\to\infty}\frac{t+2}{t+3} = \ln 1 = 0.$$

(We will use this argument lots of times on this sheet, so it is really important that you try to understand it fully.) Consequently $\int_0^\infty \frac{1}{x^2+5x+6}\,\mathrm{d}x = \ln\left(\frac{3}{2}\right)$.

(c)

$$\int_0^\infty \cos x\,\mathrm{d}x = \lim_{t\to\infty}\int_0^t \cos x\,\mathrm{d}x = \lim_{t\to\infty}\Big[\sin x\Big]_0^t = \lim_{t\to\infty}\sin t.$$

But this limit does not exist, because $\sin$ keeps oscillating between -1 and 1. Hence the integral is divergent.

(d) We have

$$\int_0^\infty e^{-x}\cos x\,\mathrm{d}x = \lim_{t\to\infty}\int_0^t e^{-x}\cos x\,\mathrm{d}x.$$

We will determine $I(t)=\int_0^t e^{-x}\cos x\,\mathrm{d}x$ using integration by parts.
Let $f(x)=e^{-x}$ and $g'(x)=\cos x$. Then $f'(x)=-e^{-x}$ and $g(x)=\sin x$. Hence

$$I(t) = \big[e^{-x}\sin x\big]_0^t + \int_0^t e^{-x}\sin x\,\mathrm{d}x.$$

Now we evaluate the second integral by integrating by parts again. Let $h(x)=e^{-x}$ and $k'(x)=\sin x$. Then $h'(x)=-e^{-x}$ and $k(x)=-\cos x$. So

$$I(t) = \big[e^{-x}\sin x\big]_0^t + \big[-e^{-x}\cos x\big]_0^t - \int_0^t e^{-x}\cos x\,\mathrm{d}x.$$

Consequently
$$2I(t) = \left[e^{-x}\sin x\right]_0^t + \left[-e^{-x}\cos x\right]_0^t$$
and so
$$I(t) = \frac{1}{2}(e^{-t}\sin t - e^{-t}\cos t + 1).$$
Now for any t, $-e^{-t} \leq e^{-t}\sin t \leq e^{-t}$ and $-e^{-t} \leq e^{-t}\cos t \leq e^{-t}$. Furthermore $\lim_{t\to\infty} e^{-t} = 0$, so we can apply the Sandwich Theorem to see that
$$\lim_{t\to\infty} e^{-t}\sin t = \lim_{t\to\infty} e^{-t}\cos t = 0.$$
Hence $\lim_{t\to\infty} I(t) = \dfrac{1}{2}$ and so
$$\int_0^\infty e^{-x}\cos x \,\mathrm{d}x = \lim_{t\to\infty} I(t) = \frac{1}{2}.$$

(e) $x^2\ln x$ is undefined at 0 but continuous on $(0,2]$. So
$$\int_0^2 x^2\ln x\,\mathrm{d}x = \lim_{t\to 0^+}\int_t^2 x^2\ln x\,\mathrm{d}x.$$
We will determine $\displaystyle\int_t^2 x^2\ln x\,\mathrm{d}x$ using integration by parts. Let $f(x) = \ln x$ and $g'(x) = x^2$. Then $f'(x) = \dfrac{1}{x}$ and $g(x) = \dfrac{x^3}{3}$. So
$$\int_t^2 x^2\ln x\,\mathrm{d}x = \left[\frac{x^3}{3}\ln x\right]_t^2 - \int_t^2 \frac{x^2}{3}\,\mathrm{d}x = \left[\frac{x^3}{3}\ln x\right]_t^2 - \left[\frac{x^3}{9}\right]_t^2 = \frac{8}{3}\ln 2 - \frac{t^3}{3}\ln t - \frac{8}{9} + \frac{t^3}{9}.$$
Using the hint we know that $\lim_{t\to 0^+}\left(\dfrac{t^3}{3}\ln t\right) = 0$ and we know that $\lim_{t\to 0^+}\dfrac{t^3}{9} = 0$. Hence
$$\int_0^2 x^2\ln x\,\mathrm{d}x = \lim_{t\to 0^+}\int_t^2 x^2\ln x\,\mathrm{d}x = \frac{8}{3}\ln 2 - \frac{8}{9}.$$

(f) $\tan x$ is undefined at $x = \pi/2$ but continuous on $[0,\pi/2)$, so
$$\int_0^{\pi/2}\tan x\,\mathrm{d}x = \lim_{t\to\pi/2^-}\int_0^t \tan x\,\mathrm{d}x = \lim_{t\to\pi/2^-}\left[\ln|\sec x|\right]_0^t = \lim_{t\to\pi/2^-}\ln|\sec t|.$$
As $t\to\pi/2^-$, $\cos t \to 0$ but $\cos t > 0$, whenever $0 \leq t < \pi/2$. Hence as $t\to\pi/2^-$, $\sec t\to\infty$ and so $\ln|\sec t|\to\infty$. Consequently the integral is divergent.

(g) $\dfrac{e^{1/x}}{x^2}$ is undefined at 0, but continuous on $(0,1]$. Hence,
$$\int_0^1 \frac{e^{1/x}}{x^2}\,\mathrm{d}x = \lim_{t\to 0^+}\int_t^1 \frac{e^{1/x}}{x^2}\,\mathrm{d}x.$$

To determine $\int_t^1 \frac{e^{1/x}}{x^2}\,\mathrm{d}x$, we substitute $u = 1/x$. $\frac{\mathrm{d}u}{\mathrm{d}x} = -\frac{1}{x^2}$, so "$\frac{1}{x^2}\,\mathrm{d}x = -\mathrm{d}u$". When $x = t$, $u = \frac{1}{t}$, and when $x = 1$, $u = 1$. Thus

$$\int_t^1 \frac{e^{1/x}}{x^2}\,\mathrm{d}x = -\int_{1/t}^1 e^u\,\mathrm{d}u = -\big[e^u\big]_{1/t}^1 = e^{1/t} - e.$$

As $t \to 0^+$, $1/t \to \infty$. Since $\exp$ is continuous, as $t \to 0^+$, $e^{1/t} \to \infty$. Consequently $\lim_{t\to 0^+} \int_t^1 \frac{e^{1/x}}{x^2}\,\mathrm{d}x = \infty$ and so the integral is divergent.

(h) The range of this integral is infinite at both ends, so following the definition of such an integral, we must split it into two integrals. We have

$$\int_{-\infty}^{\infty} \frac{1}{x^2+4}\,\mathrm{d}x = \int_{-\infty}^{0} \frac{1}{x^2+4}\,\mathrm{d}x + \int_{0}^{\infty} \frac{1}{x^2+4}\,\mathrm{d}x,$$

providing both of these integrals exist. We have

$$\int_0^{\infty} \frac{1}{x^2+4}\,\mathrm{d}x = \lim_{t\to\infty} \int_0^t \frac{1}{x^2+4}\,\mathrm{d}x = \lim_{t\to\infty} \left[\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right]_0^t = \lim_{t\to\infty} \frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right) = \frac{\pi}{4}.$$

Furthermore

$$\begin{aligned}\int_{-\infty}^{0} \frac{1}{x^2+4}\,\mathrm{d}x &= \lim_{t\to-\infty} \int_t^0 \frac{1}{x^2+4}\,\mathrm{d}x = \lim_{t\to-\infty} \left[\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right]_t^0 \\ &= \lim_{t\to-\infty} \left(-\frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right)\right) = \frac{\pi}{4}.\end{aligned}$$

Hence

$$\int_{-\infty}^{\infty} \frac{1}{x^2+4}\,\mathrm{d}x = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}.$$

(i) This integral is improper in two ways. First, the range of integration is infinite and second, $\frac{1}{\sqrt{x}(x+1)}$ is undefined at 0. We have

$$\int_0^{\infty} \frac{1}{\sqrt{x}(x+1)}\,\mathrm{d}x = \int_0^1 \frac{1}{\sqrt{x}(x+1)}\,\mathrm{d}x + \int_1^{\infty} \frac{1}{\sqrt{x}(x+1)}\,\mathrm{d}x,$$

providing both of these integrals exist.

Now

$$\int_0^1 \frac{1}{\sqrt{x}(x+1)}\,\mathrm{d}x = \lim_{t\to 0^+} \int_t^1 \frac{1}{\sqrt{x}(x+1)}\,\mathrm{d}x.$$

We evaluate $\int_t^1 \frac{1}{\sqrt{x}(x+1)}\,\mathrm{d}x$ using integration by substitution. Let $u = \sqrt{x}$. Then $\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{2\sqrt{x}}$ and so "$\frac{1}{\sqrt{x}}\,\mathrm{d}x = 2\,\mathrm{d}u$". When $x = t$, $u = \sqrt{t}$ and when $x = 1$, $u = 1$. Hence

$$\int_t^1 \frac{1}{\sqrt{x}(x+1)}\,\mathrm{d}x = 2\int_{\sqrt{t}}^1 \frac{1}{u^2+1}\,\mathrm{d}u = 2\big[\tan^{-1} u\big]_{\sqrt{t}}^1 = \pi/2 - 2\tan^{-1}\sqrt{t}.$$

As $t \to 0^+$, $\sqrt{t} \to 0$. Since $\tan^{-1}$ is continuous,

$$\lim_{t\to 0^+} \tan^{-1}\sqrt{t} = \tan^{-1}\big(\lim_{t\to 0^+}\sqrt{t}\big) = \tan^{-1} 0 = 0.$$

Hence

$$\int_0^1 \frac{1}{\sqrt{x}(x+1)}\,\mathrm{d}x = \lim_{t\to 0^+}\int_t^1 \frac{1}{\sqrt{x}(x+1)}\,\mathrm{d}x = \pi/2.$$

Similarly

$$\begin{aligned}\int_1^\infty \frac{1}{\sqrt{x}(x+1)}\,\mathrm{d}x &= \lim_{t\to\infty}\int_1^t \frac{1}{\sqrt{x}(x+1)}\,\mathrm{d}x = \lim_{t\to\infty}\big[2\tan^{-1} u\big]_1^{\sqrt{t}}\\ &= \lim_{t\to\infty} 2\tan^{-1}\sqrt{t} - \pi/2.\end{aligned}$$

As $t \to \infty$, $\sqrt{t} \to \infty$ and because $\tan^{-1}$ is continuous we have

$$\lim_{t\to\infty} \tan^{-1}(\sqrt{t}) = \lim_{u\to\infty} \tan^{-1} u = \pi/2.$$

Thus

$$\int_1^\infty \frac{1}{\sqrt{x}(x+1)}\,\mathrm{d}x = \lim_{t\to\infty}\int_1^t \frac{1}{\sqrt{x}(x+1)}\,\mathrm{d}x = \pi/2.$$

Consequently

$$\int_0^\infty \frac{1}{\sqrt{x}(x+1)}\,\mathrm{d}x = \pi/2 + \pi/2 = \pi.$$

3. How should we tackle a problem like this with a parameter? The usual idea is just to work through the problem treating the parameter as if it were a number. At some point we might see that different things will happen for different values of the parameter, so we might end up with a number of different cases.

   (a) Clearly there is no problem if $p \geq 0$. If $p < 0$ then x^p is not defined at $x = 0$. In that case

   $$\int_0^1 x^p\,\mathrm{d}x = \lim_{t\to 0^+}\int_t^1 x^p\,\mathrm{d}x.$$

   Now if $p \neq -1$,

   $$\lim_{t\to 0^+}\int_t^1 x^p\,\mathrm{d}x = \lim_{t\to 0^+}\left[\frac{x^{p+1}}{p+1}\right]_t^1 = \lim_{t\to 0^+}\left(\frac{1}{p+1} - \frac{t^{p+1}}{p+1}\right).$$

   Whether or not this limit exists depends on whether or not $\lim_{t\to 0^+} t^{p+1}$ exists. From lectures on limits we know that this limit exists (and equals zero) if and only if $p+1 \geq 0$, or equivalently if and only if $p \geq -1$. However we are not currently considering the case $p = -1$ because in that case the integral has a different form. So we know that the integral exists if $p > -1$ and does not exist if $p < -1$.
   If $p = -1$,

   $$\lim_{t\to 0^+}\int_t^1 x^p\,\mathrm{d}x = \lim_{t\to 0^+}\big[\ln x\big]_t^1 = \lim_{t\to 0^+}(-\ln t) = \infty.$$

   Hence the integral is divergent if $p = -1$.
   Consequently the integral is convergent if and only if $p > -1$ when it is equal to $\dfrac{1}{p+1}$.

(b) When $p \neq -1$ we have

$$\int_1^\infty x^p \, \mathrm{d}x = \lim_{t\to\infty} \int_1^t x^p \, \mathrm{d}x = \lim_{t\to\infty} \left[\frac{x^{p+1}}{p+1}\right]_1^t = \lim_{t\to\infty} \left(\frac{t^{p+1}}{p+1} - \frac{1}{p+1}\right).$$

Using results from lectures on limits, we know that this limit exists (and equals zero) if and only if $p+1 < 0$, or equivalently if and only if $p < -1$. Again we are not considering the case when $p = -1$ at this point, so we have shown that the integral converges if $p < -1$ and diverges if $p > -1$.

When $p = -1$,

$$\lim_{t\to\infty} \int_1^t x^p \, \mathrm{d}x = \lim_{t\to\infty} \left[\ln x\right]_1^t = \lim_{t\to\infty} \ln t = \infty.$$

Consequently the integral is convergent if and only if $p < -1$ when it is equal to $-\dfrac{1}{p+1}$.

(c) We have

$$\int_e^\infty \frac{1}{x(\ln x)^p} \, \mathrm{d}x = \lim_{t\to\infty} \int_e^t \frac{1}{x(\ln x)^p} \, \mathrm{d}x.$$

We evaluate $\displaystyle\int_e^t \frac{1}{x(\ln x)^p} \, \mathrm{d}x$ by substituting $u = \ln x$. Then $\dfrac{\mathrm{d}u}{\mathrm{d}x} = \dfrac{1}{x}$ and so "$\dfrac{1}{x}\, \mathrm{d}x = \mathrm{d}u$". When $x = e$, $u = 1$ and when $x = t$, $u = \ln t$. Hence

$$\int_e^t \frac{1}{x(\ln x)^p} \, \mathrm{d}x = \int_1^{\ln t} \frac{1}{x^p} \, \mathrm{d}x.$$

If $p \neq 1$ then

$$\int_1^{\ln t} \frac{1}{x^p} \, \mathrm{d}x = \left[\frac{x^{1-p}}{1-p}\right]_1^{\ln t} = \frac{(\ln t)^{1-p}}{1-p} - \frac{1}{1-p}.$$

As $t \to \infty$, $\ln t \to \infty$ and so $\lim_{t\to\infty} (\ln t)^{1-p}$ exists if and only if $1 - p < 0$, or equivalently if $p > 1$. So we have shown that the integral converges if $p > 1$ and diverges if $p < 1$.

If $p = 1$ then

$$\lim_{t\to\infty} \int_1^{\ln t} \frac{1}{x^p} \, \mathrm{d}x = \lim_{t\to\infty} \left[\ln x\right]_1^{\ln t} = \lim_{t\to\infty} \ln \ln t = \infty.$$

So the integral diverges if $p = 1$.

Consequently the integral converges if and only if $p > 1$, when it is equal to $\dfrac{1}{p-1}$.

4. (a) If $x \geq 1$, $x^2 \geq x$ and since $e > 1$, we have $e^{x^2} \geq e^x$. Consequently $e^{-x^2} \leq e^{-x}$.

(b) We have

$$\int_1^\infty e^{-x} \, \mathrm{d}x = \lim_{t\to\infty} \int_1^t e^{-x} \, \mathrm{d}x = \lim_{t\to\infty} \left[-e^{-x}\right]_1^t = \lim_{t\to\infty} \left(e^{-1} - e^{-t}\right) = e^{-1}.$$

(c) We can't say anything about the precise value of $\int_1^\infty e^{-x^2}\,\mathrm{d}x$, but it should seem reasonable that the integral will converge because $0 \le e^{-x^2} \le e^{-x}$ for $x \ge 1$.

5. (a) We have

$$F(s) = \int_0^\infty e^{-st}\,\mathrm{d}t = \lim_{y\to\infty}\int_0^y e^{-st}\,\mathrm{d}t = \lim_{y\to\infty}\Big[-\frac{e^{-st}}{s}\Big]_0^y = \lim_{y\to\infty}\Big(\frac{1}{s} - \frac{e^{-sy}}{s}\Big).$$

This derivation is only valid for $s \neq 0$. If $s = 0$ then we have $F(0) = \int_0^\infty 1\,\mathrm{d}t$ and it is easy to see that this is not convergent.

So for the integral to converge we need $\lim_{y\to\infty} e^{-sy}$ to exist. From our results on limits, we know that this limit exists if and only if $s \ge 0$. But $s = 0$ is already excluded for convergence and so $F(s)$ exists if and only if $s > 0$, in which case it equals $\lim_{y\to\infty}\Big(\frac{1}{s} - \frac{e^{-sy}}{s}\Big) = \frac{1}{s}$.

(b) We have

$$F(s) = \int_0^\infty e^t e^{-st}\,\mathrm{d}t = \lim_{y\to\infty}\int_0^y e^{t(1-s)}\,\mathrm{d}t = \lim_{y\to\infty}\Big[\frac{e^{t(1-s)}}{1-s}\Big]_0^y = \lim_{y\to\infty}\Big(\frac{e^{y(1-s)}}{1-s} - \frac{1}{1-s}\Big).$$

This derivation is only valid for $s \neq 1$. If $s = 1$ then we have

$$F(1) = \int_0^\infty e^t e^{-t}\,\mathrm{d}t = \int_0^\infty 1\,\mathrm{d}t$$

and this is divergent.

For this integral to converge we need $\lim_{y\to\infty} e^{y(1-s)}$ to exist. From our results on limits, we know that this limit exists if and only if $1 - s \le 0$ or equivalently $s \ge 1$. But $s = 1$ is already excluded for convergence, so $F(s)$ exists if and only if $s > 1$, in which case it equals $\lim_{y\to\infty}\Big(\frac{e^{y(1-s)}}{1-s} - \frac{1}{1-s}\Big) = \frac{1}{s-1}$.

(c) We have

$$F(s) = \int_0^\infty te^{-st}\,\mathrm{d}t = \lim_{y\to\infty}\int_0^y te^{-st}\,\mathrm{d}t.$$

When $s = 0$, $F(s) = F(0) = \int_0^\infty t\,\mathrm{d}t$ which is easily shown to be divergent.

Suppose that $s \neq 0$. We will compute $\int_0^y te^{-st}\,\mathrm{d}t$ using integration by parts. Let $g(t) = t$ and $h'(t) = e^{-st}$. Then $g'(t) = 1$ and $h(t) = -\frac{e^{-st}}{s}$. So

$$\int_0^y te^{-st}\,\mathrm{d}t = \Big[-\frac{te^{-st}}{s}\Big]_0^y + \int_0^y \frac{e^{-st}}{s}\,\mathrm{d}t = \Big[-\frac{te^{-st}}{s}\Big]_0^y - \Big[\frac{e^{-st}}{s^2}\Big]_0^y = -\frac{ye^{-sy}}{s} - \frac{e^{-sy}}{s^2} + \frac{1}{s^2}.$$

Now if $s < 0$ then $\lim_{y\to\infty}\frac{ye^{-sy}}{s^2} = \infty$ and $\lim_{y\to\infty}\frac{e^{-sy}}{s} = \infty$.

If $s > 0$ then $\lim_{y\to\infty}\Big(-\frac{ye^{-ys}}{s} - \frac{e^{-sy}}{s} + \frac{1}{s^2}\Big) = \frac{1}{s^2}$.

Hence $F(s)$ exists if and only if $s > 0$, in which case it equals $\frac{1}{s^2}$.

6. We have $e^{i\theta} = \cos\theta + i\sin\theta$ and $e^{-i\theta} = \cos\theta - i\sin\theta$. Consequently $\cos\theta = \dfrac{e^{i\theta}+e^{-i\theta}}{2}$ and $i\sin\theta = \dfrac{e^{i\theta}-e^{-i\theta}}{2}$. Now let $\theta = ix$ to get

$$\cos(ix) = \frac{e^{i^2x}+e^{-i^2x}}{2} = \frac{e^{x}+e^{-x}}{2} = \cosh x$$

and

$$-i\sin(ix) = -\frac{e^{i^2x}-e^{-i^2x}}{2} = \frac{e^{x}-e^{-x}}{2} = \sinh x.$$

7. (a) This is a standard integral, we have

$$\int_0^1 \frac{1}{x^2+4}\,\mathrm{d}x = \left[\sinh^{-1}\left(\frac{x}{2}\right)\right]_0^1 = \sinh^{-1}\left(\frac{1}{2}\right) = \ln\left(\frac{1}{2}+\sqrt{5/4}\right).$$

(b) We substitute $u = e^x$. If we do this then the denominator becomes $\sqrt{u^2+1}$ and so we can be hopeful of turning it into one of our standard integrals. We have $\dfrac{\mathrm{d}u}{\mathrm{d}x} = e^x$ and hence "$\mathrm{d}u = e^x\,\mathrm{d}x$". When $x=0$, $u=1$ and when $x=1$, $u=e$. Consequently

$$\int_0^1 \frac{e^x}{\sqrt{e^{2x}+1}}\,\mathrm{d}x = \int_1^e \frac{1}{\sqrt{1+u^2}}\,\mathrm{d}u = \left[\sinh^{-1}(u)\right]_1^e = \ln\left(\frac{e+\sqrt{e^2+1}}{1+\sqrt{2}}\right).$$

(c) The function we are integrating contains an absolute value. Our first aim should be to remove this by setting up a piecewise definition. We have

$$|x-x^2| = \begin{cases} x-x^2 & \text{if } x-x^2 \ge 0,\\ x^2-x & \text{if } x-x^2 < 0.\end{cases}$$

In order to make use of this piecewise definition, we must determine when $x-x^2 \ge 0$. First we determine when $x-x^2=0$. We have $x-x^2 = x(1-x)$ so $x-x^2=0$ if $x=0$ or $x=1$. We split up the range $[-1,2]$ into intervals bounded by the roots of $x-x^2=0$ to get the intervals $[-1,0)$, $(0,1)$ and $(1,2]$. On $[-1,0)$, $x-x^2<0$, on $(0,1)$, $x-x^2>0$ and on $(1,2]$, $x-x^2<0$. We also have $x-x^2=0$ when $x=0$ or when $x=1$. Hence

$$|x-x^2| = \begin{cases} x-x^2 & \text{if } x\in[0,1],\\ x^2-x & \text{if } x\in[-1,0)\cup(1,2].\end{cases}$$

We split the range of integration into three parts to get

$$\begin{aligned}\int_{-1}^2 |x-x^2|\,\mathrm{d}x &= \int_{-1}^0 (x^2-x)\,\mathrm{d}x + \int_0^1 (x-x^2)\,\mathrm{d}x + \int_1^2 (x^2-x)\,\mathrm{d}x\\ &= \left[\frac{x^3}{3}-\frac{x^2}{2}\right]_{-1}^0 + \left[\frac{x^2}{2}-\frac{x^3}{3}\right]_0^1 + \left[\frac{x^3}{3}-\frac{x^2}{2}\right]_1^2 = \frac{11}{6}.\end{aligned}$$

(d) We substitute $u = x^5$. Why should we do this? First, notice that the denominator of the integral is $x^{10} + 16$ so this substitution will change this to $u^2 + 16$ which is a familiar function to have in the denominator. Also the numerator is a constant times $\frac{\mathrm{d}u}{\mathrm{d}x}$ which means that it will disappear when we do the substitution. We have $\frac{\mathrm{d}u}{\mathrm{d}x} = 5x^4$ and so "$x^4\,\mathrm{d}x = \frac{1}{5}\,\mathrm{d}u$". Hence

$$\int \frac{x^4}{x^{10}+16}\,\mathrm{d}x = \frac{1}{5}\int \frac{1}{u^2+16}\,\mathrm{d}u = \frac{1}{5}\times\frac{1}{4}\tan^{-1}\left(\frac{u}{4}\right) + c = \frac{1}{20}\tan^{-1}\left(\frac{x^5}{4}\right) + c,$$

where c is a constant.

(e) We have

$$\begin{aligned}\int_0^{\pi/4} \cos^2 t\tan^2 t\,\mathrm{d}t = \int_0^{\pi/4} \sin^2 t\,\mathrm{d}t &= \int_0^{\pi/4} \frac{1}{2}(1-\cos(2t))\,\mathrm{d}t \\ &= \left[\frac{t}{2} - \frac{\sin 2t}{4}\right]_0^{\pi/4} = \frac{\pi}{8} - \frac{1}{4}.\end{aligned}$$

(f) Since we have a product where it is simple to either integrate or differentiate both parts of the product, we try to integrate by parts. Let $f(x) = \sin(4x)$ and let $g'(x) = \cos(3x)$. (Doing this the other way round should work too.) Then $f'(x) = 4\cos(4x)$ and $g(x) = \frac{\sin(3x)}{3}$ and so

$$\begin{aligned}\int_{\pi/4}^{\pi/3} \sin(4x)\cos(3x)\,\mathrm{d}x &= \left[\sin(4x)\frac{\sin 3x}{3}\right]_{\pi/4}^{\pi/3} - \frac{4}{3}\int_{\pi/4}^{\pi/3}\cos(4x)\sin(3x)\,\mathrm{d}x \\ &= -\frac{4}{3}\int_{\pi/4}^{\pi/3}\cos(4x)\sin(3x)\,\mathrm{d}x.\end{aligned}$$

Now we integrate by parts again, setting $f(x) = \cos(4x)$ and $g'(x) = \sin(3x)$. (This time we have to be careful to choose f and g' this way round, if we made the initial choice as we have done here, else we end up back where we started.) We have $f'(x) = -4\sin(4x)$ and $g(x) = -\frac{\cos(3x)}{3}$ and so

$$-\frac{4}{3}\int_{\pi/4}^{\pi/3}\cos(4x)\sin(3x)\,\mathrm{d}x = \frac{4}{3}\left[\cos(4x)\frac{\cos(3x)}{3}\right]_{\pi/4}^{\pi/3} + \frac{16}{9}\int_{\pi/4}^{\pi/3}\sin(4x)\cos(3x)\,\mathrm{d}x.$$

The final term is just the integral we started with so rearranging gives

$$\frac{7}{9}\int_{\pi/4}^{\pi/3}\sin(4x)\cos(3x)\,\mathrm{d}x = -\frac{4}{9}\left[\cos(4x)\cos(3x)\right]_{\pi/4}^{\pi/3}.$$

So

$$\int_{\pi/4}^{\pi/3}\sin(4x)\cos(3x)\,\mathrm{d}x = \frac{2}{7}(\sqrt{2}-1).$$

(g) We have $\int \frac{1}{\sqrt{x^2+4x+8}}\,\mathrm{d}x = \int \frac{1}{\sqrt{(x+2)^2+4}}\,\mathrm{d}x$. Substituting $u = x+2$ yields

$$\int \frac{1}{\sqrt{(x+2)^2+4}}\,\mathrm{d}x = \int \frac{1}{\sqrt{u^2+4}}\,\mathrm{d}u = \sinh^{-1}\left(\frac{u}{2}\right) + c = \sinh^{-1}\left(\frac{x+2}{2}\right) + c,$$

where c is a constant.

(h) We multiply the top and bottom of the fraction by e^x to get

$$\int \frac{1}{e^x - e^{-x}}\,\mathrm{d}x = \int \frac{e^x}{e^{2x}-1}\,\mathrm{d}x.$$

The denominator of the integral is a composition $f(g(x))$ where $g(x) = e^x$ and $f(x) = x^2 - 1$. So we substitute $u = e^x$. Then $\frac{\mathrm{d}u}{\mathrm{d}x} = e^x$. Hence "$e^x\,\mathrm{d}x = \mathrm{d}u$". So

$$\int \frac{e^x}{e^{2x}-1}\,\mathrm{d}x = \int \frac{1}{u^2-1}\,\mathrm{d}u.$$

In the last integral the denominator factorises as $(u-1)(u+1)$, so we can use partial fractions to compute the integral. We have

$$\frac{1}{u^2-1} = \frac{1}{(u-1)(u+1)} = \frac{A}{u-1} + \frac{B}{u+1},$$

where A and B are constants to be determined. Multiplying out gives

$$1 = A(u+1) + B(u-1).$$

By substituting $u = 1$, we get $1 = 2A$ and so $A = \frac{1}{2}$. Similarly substituting $u = -1$ gives $1 = -2B$ and so $B = -\frac{1}{2}$. Hence

$$\frac{1}{u^2-1} = \frac{1}{2(u-1)} - \frac{1}{2(u+1)}.$$

So

$$\begin{aligned}\int \frac{1}{u^2-1}\,\mathrm{d}u &= \int \left(\frac{1}{2(u-1)} - \frac{1}{2(u+1)}\right) = \frac{1}{2}\ln|u-1| - \frac{1}{2}\ln|u+1| + c \\ &= \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| + c = \frac{1}{2}\ln\left|\frac{e^x-1}{e^x+1}\right| + c,\end{aligned}$$

where c is a constant.

(i) We have

$$\begin{aligned}\int \frac{1}{\sqrt{x+1}-\sqrt{x}}\,\mathrm{d}x &= \int \frac{1}{\sqrt{x+1}-\sqrt{x}}\,\frac{\sqrt{x+1}+\sqrt{x}}{\sqrt{x+1}+\sqrt{x}}\,\mathrm{d}x = \int \frac{\sqrt{x+1}+\sqrt{x}}{x+1-x}\,\mathrm{d}x \\ &= \int \left(\sqrt{x+1}+\sqrt{x}\right)\mathrm{d}x = \frac{2}{3}(x+1)^{3/2} + \frac{2}{3}x^{3/2} + c \\ &= \frac{2}{3}\left((x+1)^{3/2} + x^{3/2}\right) + c,\end{aligned}$$

where c is a constant.

(j) One part of this function is a composition because $e^{\tan^{-1}(y)} = f(g(y))$ where $g(y) = \tan^{-1} y$ and $f(y) = e^y$. So we should try to substitute $u = \tan^{-1} y$.

We have $\frac{\mathrm{d}u}{\mathrm{d}y} = \frac{1}{1+y^2}$ and "$\mathrm{d}u = \frac{1}{1+y^2}\,\mathrm{d}y$" which looks good.

Hence

$$\int \frac{e^{\tan^{-1}(y)}}{1+y^2}\,\mathrm{d}y = \int e^u\,\mathrm{d}u = e^u + c = e^{\tan^{-1}(y)} + c,$$

where c is a constant.

(k) We have

$$\int x\sin^2 x\,\mathrm{d}x = \int \frac{x}{2}(1-\cos(2x))\,\mathrm{d}x = \int \frac{x}{2}\,\mathrm{d}x - \frac{1}{2}\int x\cos(2x)\,\mathrm{d}x.$$

We evaluate the second integral using integration by parts with $f(x) = x$ and $g'(x) = \cos(2x)$. Hence $f'(x) = 1$ and $g(x) = \frac{\sin(2x)}{2}$. Consequently

$$\int x\sin^2 x\,\mathrm{d}x = \frac{x^2}{4} - \frac{x\sin(2x)}{4} + \frac{1}{4}\int \sin(2x)\,\mathrm{d}x = \frac{x^2}{4} - \frac{x\sin(2x)}{4} - \frac{\cos(2x)}{8} + c,$$

where c is a constant.

(l) We have $\int e^{x+e^x}\,\mathrm{d}x = \int e^x e^{e^x}\,\mathrm{d}x$. Again we have a composition so we substitute for the "inner" part of the composition, i.e. we substitute $u = e^x$. Then $\frac{\mathrm{d}u}{\mathrm{d}x} = e^x$ and so "$\mathrm{d}u = e^x\,\mathrm{d}x$".

Consequently

$$\int e^{x+e^x}\,\mathrm{d}x = \int e^x e^{e^x}\,\mathrm{d}x = \int e^u\,\mathrm{d}u = e^u + c = e^{e^x} + c,$$

where c is a constant.

(m) There are lots of ways in which this integral can be written as a composition but it looks a reasonable idea to try to substitute $u = e^x$ and see what happens. We have $\frac{\mathrm{d}u}{\mathrm{d}x} = e^x$ and hence "$\mathrm{d}u = e^x\,\mathrm{d}x$" So we have

$$\int \frac{1}{1+e^x}\,\mathrm{d}x = \int \frac{e^x}{e^x(1+e^x)}\,\mathrm{d}x = \int \frac{1}{u(u+1)}\,\mathrm{d}u.$$

This last integral is found using partial fractions. We have

$$\frac{1}{u(u+1)} = \frac{A}{u} + \frac{B}{u+1}.$$

Multiplying out gives

$$1 = A(u+1) + Bu.$$

Setting $u = 0$ gives $A = 1$ and setting $u = -1$ gives $B = -1$.

Hence

$$\begin{aligned}\int \frac{1}{1+e^x}\,\mathrm{d}x &= \int \frac{1}{u(u+1)}\,\mathrm{d}u = \int \frac{1}{u}\,\mathrm{d}u - \int \frac{1}{u+1}\,\mathrm{d}u \\ &= \ln|u| - \ln|u+1| + c = x - \ln|1+e^x| + c,\end{aligned}$$

where c is a constant.

(n) We have $\displaystyle\int \ln(x^2-1)\,\mathrm{d}x = \int \ln(x-1)\,\mathrm{d}x + \int \ln(x+1)\,\mathrm{d}x$.

Now we remember how to find $\displaystyle\int \ln x\,\mathrm{d}x$ using integration by parts with $f(x) = \ln x$ and $g'(x) = 1$. We get $\displaystyle\int \ln x\,\mathrm{d}x = x\ln x - x + c$, where c is a constant.

Now return to the original integral. Substitute $u = x-1$ in the first integral and $y = x+1$ in the second integral to give

$$\begin{aligned}\int \ln(x^2-1)\,\mathrm{d}x &= \int \ln u\,\mathrm{d}u + \int \ln y\,\mathrm{d}y = u\ln u - u + y\ln y - y + c\\ &= (x-1)\ln(x-1) + (x+1)\ln(x+1) - 2x + c,\end{aligned}$$

where c is a constant.

(o) We will integrate by parts with $f(r) = \ln r$ and $g'(r) = r^4$. Then $f'(r) = \dfrac{1}{r}$ and $g(r) = \dfrac{r^5}{5}$. Hence

$$\int_1^3 r^4 \ln r\,\mathrm{d}r = \left[\frac{r^5\ln r}{5}\right]_1^3 - \int_1^3 \frac{r^4}{5}\,\mathrm{d}r = \left[\frac{r^5\ln r}{5}\right]_1^3 - \left[\frac{r^5}{25}\right]_1^3 = \frac{243\ln 3}{5} - \frac{242}{25}.$$

8. We have

$$\begin{aligned}\int_0^\infty \left(\frac{x}{x^2+1} - \frac{C}{x+1}\,\mathrm{d}x\right) &= \lim_{t\to\infty}\int_0^t \left(\frac{x}{x^2+1} - \frac{C}{x+1}\right)\mathrm{d}x = \lim_{t\to\infty}\left[\frac{1}{2}\ln(x^2+1) - C\ln|x+1|\right]_0^t\\ &= \lim_{t\to\infty}\left(\frac{1}{2}\ln(t^2+1) - C\ln(t+1)\right) = \lim_{t\to\infty}\ln\left(\frac{\sqrt{t^2+1}}{(t+1)^C}\right).\end{aligned}$$

We need to establish $\displaystyle\lim_{t\to\infty}\left(\frac{\sqrt{t^2+1}}{(t+1)^C}\right)$. If this is finite and non-zero then $\displaystyle\lim_{t\to\infty}\ln\left(\frac{\sqrt{t^2+1}}{(t+1)^C}\right)$ exists, if it is zero then $\displaystyle\lim_{t\to\infty}\ln\left(\frac{\sqrt{t^2+1}}{(t+1)^C}\right) = -\infty$ and if it is ∞ then $\displaystyle\lim_{t\to\infty}\ln\left(\frac{\sqrt{t^2+1}}{(t+1)^C}\right) = \infty$. We have

$$\frac{\sqrt{1+t^2}}{(t+1)^C} = \frac{\sqrt{1+1/t^2}}{(1+1/t)(t+1)^{C-1}} \to \begin{cases} 1 & \text{if } C = 1,\\ 0 & \text{if } C > 1, \text{ as } t\to\infty.\\ \infty & \text{if } C < 1\end{cases}$$

Hence $\displaystyle\int_0^\infty \left(\frac{x}{x^2+1} - \frac{C}{x+1}\right)\mathrm{d}x = 0$ if $C = 1$ and otherwise fails to converge.

Chapter 4

Real Series

4.1 Lecture 12: Series

Reference: Stewart Chapter 12.2, Pages 723–730. (Edition?)

In this chapter we learn about one very special type of sequence, derived from another sequence by adding together the terms one by one. This new sequence is called a **series**.

In this first release of this Chapter it is suggested that you ignore the contents of Subsection 4.1.1.

Start at Sunsection 4.1.2.

4.1.1 A Tale of a Rabbit and a Turtle following Zeno's

A snapping turtle flies the scene of its attack on a wee bonnie kit outside the cave of Caerbannog. Five minutes later, coming out of its cave, the killer rabbit, having seen its injured offspring, give chase to the turtle. Will the mighty battle between the killer rabbit and the snapping turtle take place? If so, when and where? Zeno[1] pointed out that it is possible to 'show' that the rabbit will never catch the turtle OR that the rabbit will catch the turtle. Mathematics helps to reconcile the point of view, though the final word is still debated today.

We assume that the rabbit pursue the turtle in a straight line (**1-D problem**) and that both animals are traveling at constant speed, say v_r for the rabbit, v_t for the turtle. We let the rabbit loose when the tortoise is at l_0 distance (v_t for 5') and start time when it gives chase. Using **continuous time** and **space**, if there is a meeting, we can write the following system

[1]Zeno of Elea (ca.490BC-ca.430BC) was a Greek Philosopher. He lived in the Greek 'colony' of Elea in Southern Italy (modern Velia). Although we know that his contributions to Dialectic were important, none of his writing survived intact to our times. He is best known for his paradoxes, still debated nowadays.

of equations in terms of the place of the meeting L and its time T. The rabbit will reach $L = T \cdot v_r$ and the turtle $L = l_0 + T \cdot v_t$. This is a linear system of equations,

$$\begin{aligned} L - v_r \cdot T &= 0, \\ L - v_t \cdot T &= l + 0. \end{aligned}$$

Solving the system we see that there is a unique solution

$$L = \frac{v_r l_0}{v_r - v_t}, \quad T = \frac{l_0}{v_r - v_t}.$$

Another way to look at the problem is to create a **discrete process**. So the rabbit will reach $x = l_0$ at time $t_1 = l_0/v_r$. At that time, the turtle will have traveled a further $l_1 = t_1 \cdot v_t$. At the next stage, $t_2 = l_1/v_r$ and $l_2 = t_2 \cdot v_t$, and so one. We get sequences of time $t_n = l_{n-1}/v_r$ and length $l_n = t_n \cdot v_t$. Using induction, we find

$$l_n = l_0 \cdot (v_t/v_r)^n, \quad t_n = l_0/v_r \cdot (v_t/v_r)^{n-1}, \qquad n \geq 1.$$

This process is **infinite** and a 'paradoxical' conclusion is that the rabbit will never catch the turtle. That point of view was used to discuss discrete versus continuous space and time. We leave the metaphysics aside and shall see how to reconcile both point of view using geometric series. From school, recall that $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ (we shall soon revisit that formula). Thus, we find that the total time is

$$T = (l_o/v_r) \sum_{k=0}^{\infty} (v_t/v_r)^n = \frac{l_0}{v_r} \cdot \frac{1}{1 - v_t/v_r} = \frac{l_0}{v_r - v_t}.$$

The rabbit travels

$$L_r = \sum_{k=0}^{\infty} l_k = l_0 \left(1 + \sum_{k=1}^{\infty} (v_t/v_r)^k \right) = l_0 \cdot \frac{1}{1 - v_t/v_r} = \frac{v_r l_0}{v_r - v_t},$$

and the turtle travels

$$L_t = \sum_{k=1}^{\infty} l_k = l_0 \cdot \sum_{k=1}^{\infty} (v_t/v_r)^k = \frac{v_t l_0}{v_r - v_t}.$$

Using infinite series we see that we recover the continuous time formulae.

4.1.2 Definition of a Series

If we add together the first terms of an infinite sequence $\{a_n\}_{n=1}^{\infty}$, we get another sequence $\{s_n\}_{n=1}^{\infty}$ whose terms s_n are

$$s_1 = a_1, \; s_2 = a_1 + a_2, \; s_3 = a_1 + a_2 + a_3, \; \ldots, \; s_n = a_1 + a_2 + \cdots + a_n, \; \ldots$$

This very special sort of sequence is what we will work with in this chapter.

Definition 4.1. *Given a sequence $\{a_n\}_{n=1}^{\infty}$, the corresponding* **series** *is the sequence $\{s_n\}_{n=1}^{\infty}$ of* **partial sums** *for which the n-th term is*

$$s_n = a_1 + a_2 + \cdots a_n = \sum_{k=1}^{n} a_k.$$

We denote the full series by $\sum_{k=1}^{\infty} \boldsymbol{a_k}$.

Examples.

1. The infinite sum $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots = \sum_{n=1}^{\infty} \frac{1}{n}$ is a series for the sequence $\{\frac{1}{n}\}_{n=1}^{\infty}$.

2. A series can have alternating positive and negative terms like $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = \sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n}$, or 'random' changes of sign like in $\sum_{n=1}^{\infty} \frac{\cos(n)}{n^2}$ (no π in the cosine).

4.1.3 Convergent Series, Geometric Series

We will be interested in whether the series $\sum_{k=1}^{\infty} a_k$ converges. To do this we have to determine whether the sequence $\{s_n\}_{n=1}^{\infty}$ of partial sums $s_n = \sum_{k=1}^{n} a_k$ converges.

Definition 4.2. *Given a series $\sum_{k=1}^{\infty} a_k$, let s_n denotes the n-th partial sum. If $\{s_n\}_{n=1}^{\infty}$ is convergent, the series $\sum_{k=1}^{\infty} a_k$ is* **convergent** *and the* **sum** *of the series is given by $\sum_{k=1}^{\infty} \boldsymbol{a_k} = \lim_{n\to\infty} \boldsymbol{s_n}$. If the series does not converge, it is called* **divergent**.

Remark 4.3. *If the* **value of the sum** *depends on all the terms $\{a_n\}_{n=1}^{\infty}$ involved, the* **convergence or divergence** *of the series depends only on the entries in the tail of the sequence $\{a_n\}_{n=1}^{\infty}$. For example, if $\sum_{n=100}^{\infty} a_n$ converges then $\sum_{n=1}^{\infty} a_n$ also converges with the sums related by*

$$\sum_{n=1}^{\infty} a_n = \left(\sum_{n=1}^{99} a_n\right) + \left(\sum_{n=100}^{\infty} a_n\right).$$

As we will see, series where all the terms are positive (or all the terms are negative) can be dealt with much more easily than sequences where some of the terms are positive and some are negative. Weird things can happen with these series as the following example shows.

Cautionary Tale 4.4. *Suppose we try to compute*

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\dots$$

by grouping together adjacent terms. So

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots=\left(1-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{5}-\frac{1}{6}\right)+\dots=\frac{1}{2}+\frac{1}{12}+\frac{1}{30}+\dots=\sum_{k=1}^{\infty}\frac{1}{2k(2k-1)}.$$

Alternatively since addition is commutative, we might hope to rearrange the terms to get

$$\left(1+\frac{1}{3}-\frac{1}{2}\right)+\left(\frac{1}{5}+\frac{1}{7}-\frac{1}{4}\right)+\left(\frac{1}{9}+\frac{1}{11}-\frac{1}{6}\right)+\dots=\frac{5}{6}+\frac{13}{140}+\frac{7}{198}+\dots=\sum_{k=1}^{\infty}\frac{8k-3}{2k(4k-1)(4k+3)}.$$

So, we apparently get the same sum being equal to two different series. The terms in the second sum are all bigger than the corresponding terms in the first sum because

$$\frac{8k-3}{2k(4k-1)(4k+3)}>\frac{1}{2k(2k-1)},\quad k\geq 1.$$

It can be shown that both series converge but the second one converges to a limit that is strictly greater than the first one. **This shows that if we rearrange the order of adding terms in an infinite series with positive and negative terms then we may change the value of the sum**. *When we are dealing with infinite sums, the rule that addition is commutative is no longer always valid and extreme care is required.*

We have seen that the following important series is actually one of the only 'easily' computable series.

Definition 4.5. *Let $a \neq 0$ and r be real numbers. They determine a* **geometric series (of common ratio r)** *equal to*

$$a+ar+ar^2+\dots+ar^{n-1}+\dots=\sum_{n=1}^{\infty}ar^{n-1}.$$

Note that this series if often written as $\sum_{n=0}^{\infty} ar^n$, starting at $n=0$, to simplify the notation.

Theorem 4.6. *If $|r|<1$, then*

$$\sum_{k=1}^{\infty}ar^{k-1}=\frac{a}{1-r}.$$

If $|r|\geq 1$, then $\sum_{k=1}^{\infty}ar^{k-1}$ is divergent.

Proof. The proof of this result is basically a calculation that needs to be known. When $r=1$, the series is obviously divergent. When $r\neq 1$, let $s_n=\sum_{k=1}^{n}ar^{k-1}$. Then,

$$rs_n=r\left(\sum_{k=1}^{n}ar^{k-1}\right)=\sum_{k=1}^{n}ar^k=s_n-a+ar^n=s_n-a(1-r^n),$$

so, re-arranging to s_n, $s_n=\dfrac{a(1-r^n)}{1-r}$. We can then conclude taking the limit as $n\to\infty$. □

Examples.

1. To see if the series $\sum_{k=1} 2^{2k}3^{1-k}$ converge or diverge, we can write it as the geometric series $\sum_{k=1}^{\infty} 3\left(\frac{4}{3}\right)^k$. From Theorem 4.6, the series diverges.

2. For what values of x does the series $\sum_{k=1}^{\infty} \frac{x^k}{2^k}$ converge? Here x is variable, but, again, this series is a geometric series with common ratio $\frac{x}{2}$. The series converges if and only if $\left|\frac{x}{2}\right| < 1$, i.e. if $|x| < 2$ or, equivalently, if $-2 < x < 2$. In that case its sum is $\frac{2x}{2-x}$.

4.2 Lecture 13: Important Series

Reference: Stewart Chapter 12.2, Pages 727–728, Chapter 12.3, Pages 733–736, Chapter 12.6, Pages 750–754. (Edition?)

4.2.1 A Criterion for Divergence

There is an easy test every **convergent** sequence must satisfy. This is a necessary test for convergence (or, when it fails, a **sufficient test for divergence**).

Theorem 4.7 (Divergence Criterion). *If* $\sum_{k=1}^{\infty} a_k$ *is convergent, then* $\lim_{n\to\infty} a_n = 0$*, that is, it is* **necessary** *for a convergent series that the sequence* $\{a_n\}_{n=1}^{\infty}$ *converges to* 0.

Proof. As has already been stated, $\sum_{n=1}^{\infty} a_n$ converges means that $\{s_n\}_{n=1}^{\infty}$ converges where $s_n = \sum_{k=1}^{n} a_k$. If $s = \lim_{n\to\infty} s_n$, note that $a_n = s_n - s_{n-1}$, then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (s_n - s_{n-1}) = s - s = 0.$$

□

The converse of the theorem fails. It is not necessarily true that if $\lim_{n\to\infty} a_n = 0$, then $\sum_{k=1}^{\infty} a_k$ is convergent. We shall see examples later where this happens. But it is a cheap test.

Examples.

1. The series $\sum_{n=1}^{\infty} \frac{1}{n}$, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ all satisfy the criterion, but we shall see that the first one **diverges**, but the second and third will **converge**.

2. It is not clear if the series $\sum_{n=1}^{\infty} \sin(n)$ converges or not. It sums positive and negative numbers in a random fashion. But we know it **must be divergent** because $\lim_{n\to\infty} \sin(n)$ is not determined (hence not 0).

3. In Theorem 4.6, it is clear that if $|r| \geq 1$, then the series is divergent as $\lim_{n\to\infty} |r|^n \neq 0$, and **may** be convergent when $|r| < 1$ because, in this case, $\lim_{n\to\infty} r^n = 0$.

4. Let $a_n = \frac{n+3}{n+7}$. Then $\lim_{n\to\infty} a_n = 1 \neq 0$ and so the series $\sum_{n=1}^{\infty} a_n$ diverges.

5. Let
$$a_n = (-1)^{n^3-n^2+n}\frac{n+3}{n+7}.$$
In that case, a_n changes sign (note that it changes sign like $(-1)^n$). But, anyway, a_n does not converge to 0, and so the series $\sum_{n=1}^{\infty} a_n$ diverges.

4.2.2 Telescopic Series

Let first consider an explicit example that will speak (hopefully) by itself.

Example. How can we check if the series $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$ is convergent and find its sum?

Let $s_n = \sum_{k=1}^{n} \frac{1}{k(k+1)}$. We can rewrite $\frac{1}{k(k+1)}$ using partial fractions. We have

$$\frac{1}{k(k+1)} = \frac{A}{k} + \frac{B}{k+1},$$

where A and B are constants to be determined. Multiplying out gives

$$1 = A(k+1) + Bk.$$

Setting $k = 0$ gives $1 = A$. Setting $k = -1$ gives $1 = -B$ so $B = -1$. Hence

$$\frac{1}{k(k+1)} = \frac{1}{k} + \frac{-1}{k+1}.$$

So,

$$\begin{aligned} s_n &= \sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) \\ &= 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{1}{n-1} - \frac{1}{n} + \frac{1}{n} - \frac{1}{n+1} = 1 - \frac{1}{n+1}, \end{aligned}$$

hence $\lim_{n\to\infty} s_n = 1$.

This sort of sum is an example of what is called a telescoping sum because its terms cancel each other, collapsing into almost nothing.

Definition 4.8. *A finite sum $s_n = \sum_{k=1}^{n} a_k$ in which subsequent terms cancel each other, leaving only the initial and final terms is called a* **telescopic sum**.

Examples.

1. Given a finite set $\{a_k\}_{k=1}^{n}$ of real numbers, the sum

   $$s_n = \sum_{k=1}^{n-1} (a_i - a_{i+1}) = a_1 - a_n$$

   is telescopic.

2. We have seen before that the series $\sum_{n=1}^{\infty} \sin(n)$ is divergent. Remarkably we can use telescoping sums to show such result. Using trigonometric identities, we have

$$\begin{aligned}\sum_{n=1}^{N} \sin(n) &= \sum_{n=1}^{N} \frac{1}{2}\csc(1/2)\left(2\sin(1/2)\sin(n)\right)\\ &= \frac{1}{2}\csc(1/2)\sum_{n=1}^{N}\left(\cos(\frac{2n-1}{2}) - \cos(\frac{2n+1}{2})\right)\\ &= \frac{1}{2}\csc(1/2)\left(\cos(\frac{1}{2}) - \cos(\frac{2N+1}{2})\right).\end{aligned}$$

We can then see that such sum has no limit as $N \to \infty$.

4.2.3 Harmonic Series

Theorem 4.9. *The* **Harmonic Series** $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$ *diverges.*

Proof. Let $a_n = \frac{1}{n}$. Consider the following inequalities

$$\begin{aligned} a_1 &= 1 \geq 1,\\ a_1 + a_2 &= 1 + \frac{1}{2} \geq 1 + \frac{1}{2},\\ a_1 + a_2 + a_3 + a_4 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \geq 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{2}{2},\\ a_1 + \cdots + a_8 &= 1 + \frac{1}{2} + \ldots + \frac{1}{7} + \frac{1}{8} \geq 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) = 1 + \frac{3}{2},\\ \cdots &\geq \cdots,\\ a_1 + \cdots + a_{2^n} &\geq 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \cdots + \left(\frac{1}{2^n} + \cdots + \frac{1}{2^n}\right) = 1 + \frac{n}{2}.\end{aligned}$$

This shows that we can make $\sum_{k=1}^{n} \frac{1}{k}$ as large as we like by choosing n large enough. Consequently the series cannot converge. □

Cautionary Tale 4.10. *Note that, even though* $\lim_{n\to\infty} a_n = 0$, *this series* **diverges**. *This is the standard example of a series that behaves in this way.*

A similar technique to the above can be used to show that the series of the inverse of squares of integers converges.

Lemma 4.11. *The series* $\sum_{n=1}^{\infty} 1/n^2$ *converges and its sum is less than 2.*

Proof. Let

$$\begin{aligned} t_n = \sum_{k=1}^{2^{(n+1)}-1} \frac{1}{k^2} &= 1 + \left(\frac{1}{2^2} + \frac{1}{3^2}\right) + \left(\frac{1}{4^2} + \cdots + \frac{1}{7^2}\right) + \cdots + \left(\frac{1}{2^{2n}} + \cdots + \frac{1}{(2^{(n+1)} - 1)^2}\right) \\ &\leqslant 1 + \frac{2}{2^2} + \frac{4}{4^2} - \left(\frac{1}{4} - \frac{1}{9}\right) + \cdots + \frac{2^n}{2^{2n}} \\ &= 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} - \left(\frac{5}{36}\right) \\ &= \frac{1 - (1/2)^{(n+1)}}{1 - (1/2)} - \left(\frac{5}{36}\right) = \frac{67}{36} - \frac{1}{2^n} < 2, \end{aligned}$$

where in the above we have used the formula for summing a geometric series. The sequence of partial sums is hence increasing and bounded above by $\frac{67}{36} \approx 1.86\overline{1}$ and so, by the monotone convergence theorem it converges to a limit smaller than 2. □

Remark 4.12. *We shall see that, as a consequence of our analysis of Fourier series (see MA2712), we can calculate the sum of the series:*

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \approx 1.645 \ldots < 2.$$

4.2.4 Algebra of Series

We now see that we can combine series like we have combined functions and sequences etc. However, there are not so many ways in which we can combine series.

Theorem 4.13 (Algebra of Limits for Series)**.** *Suppose* $\sum_{k=1}^{\infty} a_k$ *converges to* a *and* $\sum_{k=1}^{\infty} b_k$ *converges to* b *and* $c \in \mathbb{R}$ *is a constant. Then,*

1. $\sum_{k=1}^{\infty} (a_k + b_k)$ *converges to* $a + b$*;*

2. $\sum_{k=1}^{\infty} (a_k - b_k)$ *converges to* $a - b$*;*

3. $\sum_{k=1}^{\infty} c a_k$ *converges to* ca*.*

Proof. We will prove the first part. The other parts are left as Exercise 7 of Sheet 3a.

Let us define the sequences $\{s_n\}_{n=1}^{\infty}$, $\{t_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ by

$$s_n = \sum_{k=1}^{n} a_k, \qquad t_n = \sum_{k=1}^{n} b_k, \qquad u_n = \sum_{k=1}^{n} (a_k + b_k).$$

Then $u_n = s_n + t_n$. Since $\lim_{n\to\infty} s_n = a$ and $\lim_{n\to\infty} t_n = b$, using Theorem 4.13, we have

$$\lim_{n\to\infty}(s_n + t_n) = \lim_{n\to\infty} s_n + \lim_{n\to\infty} t_n = a + b.$$

Consequently,

$$\sum_{k=1}^{\infty}(a_k + b_k) = \lim_{n\to\infty} u_n = \lim_{n\to\infty}(s_n + t_n) = a + b.$$

Therefore $\sum_{k=1}^{\infty}(a_k + b_k)$ converges to $a + b$. □

This result will be used to split complicated formula for a_k into manageable parts.

4.3 Lecture 14: Test for Convergence

Reference: Stewart Chapter 12.3, Pages 733-739, Chapter 12.4, Pages 741-744 and Chapter 12.6, Pages 752-754. (Edition?)

Many series are impossible to sum exactly. Second best is to determine its convergence (or divergence). There are tests that allow us to determine whether a series converges, but they will not tell us anything about their sum. We shall see in Chapter 7 that many functions can be represented by a series of monomial terms. Knowing that a series converges will allow us to perform some manipulations on the terms of the series and hence on the function. For instance, we should be able to integrate and differentiate these series and so to solve some differential equations (see MA3610 in Level 3). We will look at four tests. Some tests may be inconclusive on a given series. In that case we should use them one by one until a conclusion is reached (all four might fail though).

Before we look into those tests, we prove an important lemma. Recall that bounded monotone sequences converge. If the terms are positive, the **sequence of partial sums** of a series must be **increasing**, and so **if we can bound them a priori**, **the series must be convergent**. This leads to the following useful result where we can ignore the change of signs of some series.

Lemma 4.14. *Suppose that the series* $\sum_{n=1}^{\infty} |a_n|$ *converges. Then, the series* $\sum_{n=1}^{\infty} a_n$ *converges.*

Proof. Let $b_k = |a_k| + a_k$. Note that $0 \leq b_k \leq 2|a_k|$. Let $s_n = \sum_{k=1}^{n} b_k$. Then, $\{s_n\}_{n=1}^{\infty}$ is an increasing sequence (because $b_k \geq 0$) and bounded because

$$0 \leq s_n = \sum_{k=1}^{n} b_k \leq 2\sum_{k=1}^{n} |a_k| \leq 2\sum_{k=1}^{\infty} |a_k|.$$

Consequently, from Theorem 2.23, it converges. Now, $a_k = b_k - |a_k|$ and both $\sum_{k=1}^{\infty} b_k$ and $\sum_{k=1}^{\infty} |a_k|$ converge. Therefore $\sum_{k=1}^{\infty} a_k$ converges from Theorem 4.13. □

Remark 4.15. *We shall see in Section 5.3 that a convergent series* $\sum_{n=1}^{\infty} a_n$ *such that* $\sum_{n=1}^{\infty} |a_n|$ *does not converge behaves quite extraordinarily.*

Examples.

1. The series $\sum_{n=1}^{\infty} \frac{\cos(n)}{n^2}$ converges because $\left|\frac{\cos(n)}{n^2}\right| \leq \frac{1}{n^2}$ and the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (Lemma 4.14 and the Comparison Test). But its **sum is difficult to calculate**.

2. We cannot apply Lemma 4.14 to $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n}$ because the harmonic series diverge. This follows because $\cos(n\pi) = (-1)^n$ and so $\left|\frac{\cos(n\pi)}{n}\right| = \frac{1}{n}$. We shall see that this series has some interesting, although unexpected, properties.

4.3.1 Comparison Test

Finally, we can use our results about monotone sequences to give the following existence result.

Proposition 4.16. *Suppose that $\{a_n\}_{n=1}^{\infty}$ is a sequence of* **positive** *numbers, then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the sequence of partial sums is bounded.*

Proof. If the terms of the series are positive, its sequence of partial sums is monotonically increasing, and so, from the Monotone Convergence Theorem 2.23, it has a limit if and only if it is bounded. □

Given the above comments we restrict attention to non-negative sequences $\{a_n\}_{n=1}^{\infty}$. Hence we establish convergence or divergence by testing whether or not **the sequence of partial sums $\{s_n\}_{n=1}^{\infty}$ is bounded**. This is usually done by comparing the given series with a 'simpler' series which is known to converge or diverge. We make this precise in the following result.

Theorem 4.17 (Comparison Test)**.** *Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be real sequences of numbers. Suppose that there exists an $N > 0$ such that $0 \leqslant a_n \leqslant b_n$ for all $n \geq N$. Then,*

(i) if $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges,

(ii) if $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

Proof. Recall that convergence or divergence do not depend on the **first N terms**. Without limiting the generality we can assume that $N = 1$. If $\sum_{n=1}^{\infty} b_n$ converges then the sequence of partial sums of $\sum_{n=1}^{\infty} a_n$ is bounded, hence it converges. If $\sum_{n=1}^{\infty} a_n$ diverges then the sequence of partial sums of $\{a_n\}_{n=1}^{\infty}$ is unbounded. From the inequality the sequence of partial sums of $\{b_n\}_{n=1}^{\infty}$ is also unbounded and hence $\sum_{n=1}^{\infty} b_n$ diverges. □

Examples.

1. Let $a_n = \dfrac{n+3}{n^2+7}$. We can compare the series $\displaystyle\sum_{n\to\infty} a_n$ with the harmonic series because $a_n \geq \dfrac{1}{n}$ for $n \geq 3$. Hence the series $\displaystyle\sum_{n\to\infty} a_n$ diverges because the harmonic series diverges.

2. Let now $a_n = \dfrac{n+3}{n^3+7}$. We cannot compare a_n with $\dfrac{1}{n^2}$ directly, but $a_n \leq \dfrac{2}{n^2}$ for $n \geq 3$. And so the series $\displaystyle\sum_{n\to\infty} a_n$ converges by comparison with $\sum_{n=1}^{\infty}(2/n^2)$.

The next result is an equivalent test, but often easier to use.

Theorem 4.18 (Limit Comparison Test)**.** *Suppose that $\displaystyle\sum_{n=1}^{\infty} a_n$ and $\displaystyle\sum_{n=1}^{\infty} b_n$ are series of positive terms and $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = l$ where $l > 0$ (and is finite). Then, either both series converge, or both diverge together.*

This test follows from Lemma 4.14 and Theorem 2.23. We shall look at its proof in Chapter 5.

4.3.2 Integral Test

We now consider a test which can be used for series of the type

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

for fixed $p > 0$. When $p \le 1$, we see that $n^p \le n$, $\dfrac{1}{n} \le \dfrac{1}{n^p}$, and so the series diverges using the Comparison Test. When $p \ge 2$, we have $n^2 \le n^p$, $\dfrac{1}{n^p} \le \dfrac{1}{n^2}$, and so the series converges using the Comparison Test. **When $1 < p < 2$ we cannot conclude**. We need another idea. The next test that we will consider applies to series of the form $\sum_{n=1}^{\infty} f(n)$ where $f : [1, \infty) \to [0, \infty)$ is a positive decreasing continuous function with $f(x) \to 0$ as $x \to \infty$. In the above case the function $f(x) = 1/x^p$ has this property.

Theorem 4.19 (Integral Test)**.** *If $f : [1, \infty) \to [0, \infty)$ is a positive decreasing positive continuous function then the following quantities*

$$s = \sum_{n=1}^{\infty} f(n) \quad \textit{and} \quad I = \int_1^{\infty} f(x)\, dx$$

either both converge or both diverge.

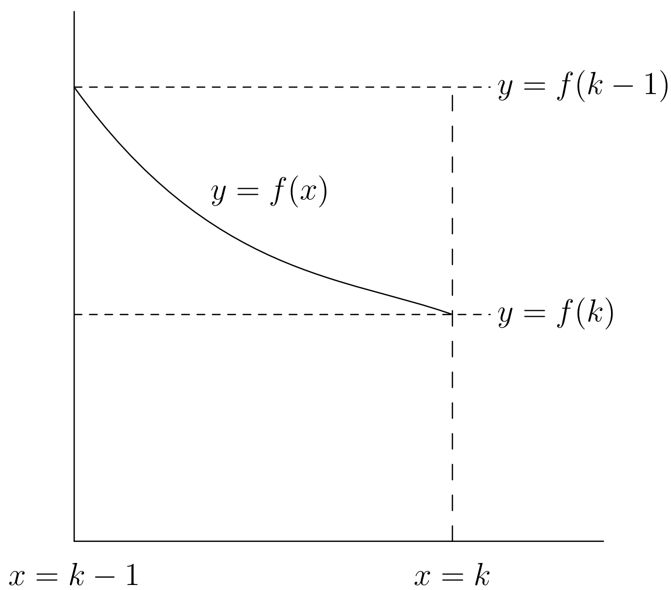


Figure 4.1: The area under the curve is less than the area under the line $y = f(k-1)$ and it is greater than the area under the line $y = f(k)$.

Proof. If f is decreasing then, for $x \in [k-1, k]$, we have

$$a_k = f(k) \leqslant f(x) \leqslant f(k-1) = a_{k-1}.$$

By considering the area under the curves of $y = f(k)$, $y = f(x)$ and $y = f(k-1)$, when $k-1 \leqslant x \leqslant k$, we have

$$f(k) = \int_{k-1}^{k} f(k)\,dx \leqslant \int_{k-1}^{k} f(x)\,dx \leqslant \int_{k-1}^{k} f(k-1)\,dx = f(k-1),$$

see Figure 4.1. Note that both $\{s_n\}_{n=1}^{\infty}$ and $\{I_n\}_{n=1}^{\infty}$ are increasing sequences and

$$s_n - f(1) = \sum_{k=2}^{n} f(k) \leqslant \left(\int_1^2 + \int_2^3 + \cdots + \int_{n-1}^{n}\right) f(x)\,dx \leqslant \sum_{k=2}^{n} f(k-1) = \sum_{k=1}^{n-1} f(k) = s_{n-1}.$$

That is,

$$s_n - f(1) \leqslant I_n \leqslant s_{n-1}.$$

Hence, if $\{I_n\}_{n=1}^{\infty}$ converges then $\{s_n\}_{n=1}^{\infty}$ converges and if $\{I_n\}_{n=1}^{\infty}$ diverges then $\{s_n\}_{n=1}^{\infty}$ diverges from the Comparison Test. □

Remarks 4.20. *As before, the convergent result will remain true if the hypothesis are true for n, x on an interval $[N, \infty)$ for some N large enough (see Exercise 6 of Sheet 3a). Moreover, integrating functions can be very difficult but, using the Limit Comparison Test, we can simplify $f(n)$ into an expression that can be integrated.*

Example. Check for which $p > 0$, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent. For the Integral Test, let $a_n = \dfrac{1}{n^p}$ and let $f : [1, \infty) \to \mathbb{R}$, $f(x) = \dfrac{1}{x^p}$. Then f is continuous on $[1, \infty)$, positive and decreasing. Furthermore $a_n = f(n)$ and so we may apply the test. We have

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^p}\,dx = \lim_{t\to\infty} \begin{cases} t^{1-p}/(1-p), & 0 < p < 1; \\ \ln(t), & p = 1; \\ \frac{1}{p-1}\left(1 - \frac{1}{t^{p-1}}\right), & p > 1. \end{cases}$$

The first two limit are infinity, but the last one is 0. Consequently the integral, and hence the series, is divergent when $p \leq 1$ and is convergent when $p > 1$.

The next test can be used for series of the type $\displaystyle\sum_{n=1}^{\infty} f(n)$ where f is integrable basically on the positive real line.

Cautionary Tale 4.21. *Note that the sum of the series is not necessarily equal to the improper integral. After all, the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ starts $1 + \frac{1}{4} + \frac{1}{9} + \cdots$ and consists of positive terms, so its sum is larger than 1.5 (suming the first 10 terms). But the integral test gives*

$$\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx = \lim_{t\to\infty} \left[-\frac{1}{x}\right]_1^t = \lim_{t\to\infty} \left(1 - \frac{1}{t}\right) = 1.$$

4.4 Lecture 15: Further Tests for Convergence

4.4.1 Comparing a Series with a Geometric Series: the Root and Ratio Tests

There are two tests for convergence involving comparing a given series with a geometric series. First a theorem that is mainly due to d'Alembert[2].

Theorem 4.22 (d'Alembert's Ratio Test). *Let $\{a_n\}_{n=1}^{\infty}$ be a real sequence such that*

$$\lim_{n\to\infty} |a_{n+1}/a_n| = \rho_1.$$

1. *If $\rho_1 < 1$, the series $\sum_{n=1}^{\infty} a_n$ converges.*
2. *If $\rho_1 > 1$, the series $\sum_{n=1}^{\infty} a_n$ diverges.*
3. *If $\rho_1 = 1$, we cannot conclude.*

The second test is mainly due to Cauchy [3].

Theorem 4.23 (Cauchy's Root Test). *Let $\{a_n\}_{n=1}^{\infty}$ be a real sequence such that*

$$\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \rho_2.$$

1. *If $\rho_2 < 1$, the series $\sum_{n=1}^{\infty} a_n$ converges.*
2. *If $\rho_2 > 1$, the series $\sum_{n=1}^{\infty} a_n$ diverges.*
3. *If $\rho = 1$, we cannot conclude.*

Those two results are corollaries of the following lemma about the convergence/divergence of series with general terms.

Lemma 4.24. *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of terms such that there exists an N with,*

[2] J.L.R. d'Alembert(1717-1783) was a French mathematician. He was at the forefront of the mathematics of his time, in particular in calculus and mathematical methods for physics where he was one of the first to use PDEs. He worked on the *Encyclopédie*. He was a difficult man and fell out with many contemporaries. At the end of his life, he devoted more time to philosophy and literature and became the Perpetual Secretary of the *Académie Française*.

[3] A.L. Cauchy (1789-1857) was a French mathematician who made analysis what it is today via his work on convergence. The first analysis course mathematician are following today are still broadly based on his own at the École Polytechnique in 1821. Cauchy was also very productive in all areas of analysis and differential equations. His full life work covers 27 volumes.

1. either

$$|a_{n+1}/a_n| \leq \rho_1 < 1, \quad n \geq N,$$

or

$$|a_n|^{1/n} \leq \rho_2 < 1, \quad n \geq N,$$

then the series converges.

2. If, either

$$|a_{n+1}/a_n| \geq \rho_1 > 1, \quad n \geq N,$$

or

$$|a_n|^{1/n} \geq \rho_2 > 1, \quad n \geq N,$$

then the series diverges.

Proof. This proof has been removed. It had a flaw. See the lecture and seminar notes.

□

Example. We can determine the values of x for which $\sum_{n=1}^{\infty} \frac{x^n}{n!}$ is convergent. If the sum depends on a variable x like this one does, it is often a good idea to use the Ratio Test. We have $a_n = \frac{x^n}{n!}$ and we check the hypotheses of the test. First, we need a_n to be non-zero and this is ok providing $x \neq 0$, so let's assume $x \neq 0$ for the moment. Second, $\frac{a_{n+1}}{a_n} = \frac{x^{n+1}/(n+1)!}{x^n/n!} = \frac{x}{n+1}$. So, we have

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{|x|}{n+1} = 0.$$

Using the Ratio Test, we see that the series converges providing $x \neq 0$. But the case when $x = 0$ is easy because then every term of the series is zero and so it converges. Consequently the series converges for all x.

Cautionary Tale 4.25. *Notice that if $l = 1$ then the Ratio Test tells us nothing. Typical examples are when we have rational functions (i.e. ratio of polynomials).*

Example. Check for which $p > 0$, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent applying the last two tests.

1. If we try to use the Ratio Test, we have

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n^p}{(n+1)^p} = \left(\frac{n}{(n+1)}\right)^p.$$

   Hence, because p is a constant (independent of n), the Ratio Test is inconclusive:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \left(\lim_{n\to\infty} \frac{a_{n+1}}{a_n}\right)^p = 1^p = 1.$$

2. For the Root Test, consider

$$b_n = \ln(|a_n|^{1/n}) = \frac{1}{n}\ln(n^{-p}) = -p\,\frac{\ln n}{n},$$

and so $\lim_{n\to\infty} b_n = 0$ and the test is inconclusive because

$$\lim_{n\to\infty} |a_n|^{1/n} = \exp(\lim_{n\to\infty} b_n) = 1.$$

Remark 4.26. *How do we know whether one of these tests will tell us about convergence? We don't for sure. However, if the general term of the series looks like something we could integrate, then we should try the integral test. On the other hand, if the ratio of consecutive terms of the series is simple enough, we should try the ratio test.*

Summary of Chapter 3

We have seen:

- the definition of a **series**;
- the **geometric** and **harmonic** series;
- how to sum **telescoping sums**;
- an example of a series where the **terms tend to zero but the series diverges**;
- **four tests** for determining whether a series is convergent;
- that we must **check carefully** that it is appropriate to use either of the tests;
- that the series $\sum_{k=1}^{\infty} \frac{1}{n^p}$, $p \geq 0$, converges for $p > 1$ and diverges for $p \leq 1$.

4.5 Exercise Sheet 3

4.5.1 Exercise Sheet 3a

1. Determine whether the following series converge. If they do, find the limit.
   (a) $\displaystyle\sum_{k=1}^{\infty}\frac{3^k}{2^{3k}}$. (b) $\displaystyle\sum_{k=1}^{\infty}\frac{1}{(k+1)(k+2)}$. (c) $\displaystyle\sum_{k=1}^{\infty}(-1)^{3k}$. (d) $\displaystyle\sum_{k=3}^{\infty}\frac{3}{2^k}$.

2. In each of the following cases determine whether or not the series converges.
   (a) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2^n+1}$. (b) $\displaystyle\sum_{n=1}^{\infty}\frac{4n^2-n+3}{n^3+2n}$. (c) $\displaystyle\sum_{n=1}^{\infty}\frac{n+\sqrt{n}}{2n^3-1}$. (d) $\displaystyle\sum_{n=1}^{\infty}n^4\mathrm{e}^{-n^2}$.

3. For each of the following series, determine the values of $x \in \mathbb{R}$ such that the given series converges.

   (a) $\displaystyle\sum_{k=0}^{\infty}\frac{x^k}{k!}$.

   (b) In the following $\alpha \in \mathbb{R}$ is not an integer.
   $$\sum_{k=0}^{\infty}\left(\frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}\right)x^k = 1+\alpha x+\frac{\alpha(\alpha-1)}{2}x^2+\cdots$$

   (c) $\displaystyle\sum_{k=0}^{\infty}\frac{k^3x^k}{3^k}$.

   (d) $\displaystyle\sum_{k=0}^{\infty}k^kx^k$.

   (e) $\displaystyle\sum_{k=0}^{\infty}a_kx^k = 1+2x+x^2+2x^3+x^4+\cdots$, i.e. with $a_{2k}=1$ and $a_{2k+1}=2$ for $k=0,1,2,\cdots$.

   (f) $\displaystyle\sum_{k=1}^{\infty}\frac{\sqrt{x^2+k}-|x|}{k^2}$.

   (g) $\displaystyle\sum_{k=1}^{\infty}\left(\frac{\cos kx}{k^3}+3\frac{\sin kx}{k^2}\right)$.

4. Show that
   $$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\cdots=\frac{1}{2}.$$

5. Use the Integral Test to test the convergence of the following series for $c>0$:
   (a) $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^c}$. (b) $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n(\ln n)(\ln\ln n)^c}$.

6. Show that the Integral Test remain valid if its hypotheses on f are only true for some interval $[N,\infty)$, $N \in \mathbb{N}$.

7. Prove Parts 2 and 3 of Theorem 4.13.

8. Determine the values of x for which the following series converge. Find the sum of the series for those values of x. (a) $\sum_{k=1}^{\infty} \frac{x^k}{3^k}$. (b) $\sum_{k=1}^{\infty}(x-4)^k$. (c) $\sum_{k=1}^{\infty} \tan^k x$.

9. Show that the Ratio Test is always inconclusive for series whose terms are rational functions, that is a ratio of polynomials $a_n = \frac{p(n)}{q(n)}$ for p, q some polynomials. Deduce by other means the convergence of the series $\sum_{n=1}^{\infty} a_n$.

10. Get the details of the proof of the Root Test when $\rho_2 < 1$.

4.5.2 Additional Exercise Sheet 3b

1. Determine whether or not the following series are convergent.
(a) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+2}}$; (b) $\sum_{n=1}^{\infty} \frac{n}{e^n}$; (c) $\sum_{n=1}^{\infty} \frac{n+4}{n+1}$; (d) $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$; (e) $\sum_{n=1}^{\infty} \frac{(-3)^n}{n!}$.

2. Determine which of the following series converge and for those series that do converge find the sum.
(a) $\sum_{k=1}^{\infty} \frac{3}{5^k}$; (b) $\sum_{k=1}^{\infty} \frac{1}{(k+2)(k+3)}$; (c) $\sum_{k=1}^{\infty} \frac{1}{9k^2+3k-2}$; (d) $\sum_{k=1}^{\infty} \frac{k^2+2}{k^3+3}$; (e) $\sum_{k=1}^{\infty} \frac{3^k+2^k}{6^k}$.

3. Determine whether $\sum_{n=1}^{\infty} \frac{7(n+1)}{2^n n}$ converges or diverges.

4. Determine whether $\sum_{n=1}^{\infty} \frac{n^4+3n^2}{n^{9/2}+n}$ converges or diverges.

Short Feedback for Sheet 3a

1. (a) 3/5.
   (b) 1/2.
   (c) divergent.
   (d) 3/4.

2. (a) convergent.
   (b) divergent.
   (c) convergent.
   (d) convergent.

3. (a) $x \in \mathbb{R}$.
   (b) $|x| < 1$.
   (c) $|x| < 3$.

   (d) $x = 0$.
   (e) $|x| < 1$.
   (f) $x \in \mathbb{R}$. Compare with a power of k.
   (g) $x \in \mathbb{R}$.

4. The sum is telescopic.

5. (a) convergent when $c > 1$.
   (b) convergent when $c > 1$.

6. The first $N - 1$ terms do not influence convergence of the full series.

7. Use the algebra of the limits of sequences.

8. (a) Converges when $|x| < 3$ to $\dfrac{x}{3-x}$.
   (b) Converges when $3 < x < 5$ to $\dfrac{x-4}{5-x}$.
   (c) Converges when $x \neq \pm\dfrac{\pi}{4} + k\pi$, $k \in \mathbb{Z}$, to $\dfrac{\tan(x)}{1-\tan(x)}$.

9. Show that for any polynomial $p(x)$, $\lim_{n\to\infty} \dfrac{p(n+1)}{p(n)} = 1$ and use the Comparison or Integral Tests.

10. Use a similar comparison than for the Ratio Test.

Short Feedback for the Additional Exercise Sheet 3b

1. (a) Divergent;
   (b) convergent;
   (c) divergent;
   (d) divergent;
   (e) convergent.

2. (a) 3/4;
   (b) telescopic;
   (c) telescopic;
   (d) divergent;
   (e) 3/2.

3.

4.5.3 Feedback for Sheet 3a

1. It is a geometric series of ratio 3/8 and so it converges to its limit $\dfrac{3/8}{1-3/8} = \dfrac{3}{5}$.

   We have a telescopic series. We decompose

$$\frac{1}{(k+1)(k+2)} = \frac{a}{k+1} + \frac{b}{k+2} = \frac{1}{k+1} - \frac{1}{k+2},$$

   because $a(k+2) + b(k+1) = 1$ for all $k \in \mathbb{N}$. Therefore the terms from $n = 3$ cancel out and so the sum is 1/2.

   (b) The series is divergent because $(-1)^k$ does not converge to 0.

   (c) We have $\displaystyle\sum_{k=3}^{\infty} \frac{3}{2^k} = \frac{3}{8} \cdot \sum_{k=0}^{\infty} \frac{1}{2^k} = \frac{3}{8} \cdot \frac{1/2}{1-1/2} = \frac{3}{4}$.

2. (a) We could show convergence here by using the Ratio or Root Tests or more simply by using the Comparison Test by noting that

$$0 \leqslant \frac{1}{2^n+1} \leqslant \frac{1}{2^n}.$$

   The upper bound is a term from a convergent geometric series.

   (b) This is divergent. Using the Limit Comparison Test with $a_n = \dfrac{4n^2 - n + 3}{n^3 + 2n}$ and $b_n = 1/n$, we get

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} n a_n = \lim_{n\to\infty} \frac{n(4n^2 - n + 3)}{n^3 + 2n} = 4,$$

   and so both series diverge together.

   (c) This converges. Using the simple Comparison Theorem we see that

$$\frac{n + \sqrt{n}}{2n^3 - 1} \leq \frac{2n}{2n^3 - 1}$$

   and, using the Limit Comparison Test with $a_n = \dfrac{2n}{2n^3 - 1}$ and $b_n = 1/n^2$, we get

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} n^2 a_n = \lim_{n\to\infty} \frac{2n^3}{2n^3 - 1} = 1,$$

   and so all series converge together.

   (d) Using the Root Test with $a_n = n^4 e^{-n^2}$, we have

$$\lim_{n\to\infty} a_n^{1/n} = \lim_{n\to\infty} (n^{1/n})^4 \left(e^{-n^2}\right)^{1/n} = \lim_{n\to\infty} (n^{1/n})^4 e^{-n} = 0.$$

   Here the results is as a consequence of $\lim_{n\to\infty} n^{1/n} = 1$ and $\lim_{n\to\infty} e^{-n} = 0$. And so, the series converges.

3. (a) Let $a_k = x^k/k!$ and use the Ratio Test. We have

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\frac{|x|^{k+1}/(k+1)!}{|x|^k/k!} = \lim_{k\to\infty}\frac{|x|}{k+1} = 0.$$

From the Ratio Test, the series actually converges for all $x \in \mathbb{R}$.

(b) Let $a_k = \alpha(\alpha-1)\cdots(\alpha-k+1)x^k/k!$. Using the Ratio Test

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\frac{\alpha-k}{k+1}|x| = \lim_{k\to\infty}\frac{\alpha/k-1}{1+1/k}|x| = |x|.$$

Thus the series converges if $|x| < 1$. If $|x| > 1$ then the terms of the series are unbounded and thus the series diverges. What happens when $x = -1$ or $x = 1$ needs more refined tests to determine if the series converges or diverges and the outcome depends on α. This will not be considered further here.

(c) The Root Test is the easiest test to use here. With $a_k = k^3x^k/3^k$ we have

$$\lim_{k\to\infty}|a_k|^{1/k} = \lim_{k\to\infty}\frac{(k^{1/k})^3|x|}{3} = \frac{|x|}{3}.$$

From the Root Test, the series converges if $|x| < 3$. It diverges if $|x| > 3$. If $|x| = 3$ then $|a_k| = k^3$ and since these terms become unbounded, it follows that the series diverges when $x = \pm 3$.

(d) The Root Test is the easiest test to use here. With $a_k = k^kx^k$, we have

$$|a_k^{1/k}| = |kx|.$$

This only converges if $x = 0$ and is unbounded for $x \neq 0$. Hence the series only converges when $x = 0$.

(e) Let $b_k = a_kx^k$. The Ratio Test does not give any information here as a_{k+1}/a_k does not have a limit as $k \to \infty$. However we can still use the Root Test. Since

$$1 \leqslant a_k \leqslant 2, \quad 1 \leqslant \lim_{k\to\infty} a_k^{1/k} \leqslant \lim_{k\to\infty} 2^{1/k} = 1.$$

Thus,

$$\lim_{k\to\infty}|b_k|^{1/k} = \lim_{k\to\infty} a_k^{1/k}|x| = |x|.$$

The series converges if $|x| < 1$ and diverges if $|x| > 1$. By inspection the series diverges if $x = 1$ as the terms of the series do not tend to 0 as $k \to \infty$. It can be shown that the series also diverges when $x = -1$.

(f) Since $x^2 \geq 0$,

$$a_k = \frac{\sqrt{x^2+k}-|x|}{k^2} = \frac{(x^2+k)-x^2}{(\sqrt{x^2+k}+|x|)k^2} = \frac{1}{(\sqrt{x^2+k}+|x|)k} \leqslant \frac{1}{k^{3/2}},$$

for all $x \in \mathbb{R}$. Since $0 \leqslant a_k \leqslant 1/k^{3/2}$, the series $\sum_{k=1}^{\infty} a_k$ converges by comparison with the convergent series $\sum_{k=1}^{\infty} k^{-3/2}$ for all $x \in \mathbb{R}$.

(g) Let a_k denotes the k-th term of the series. From the triangle inequality and the fact that $|\sin kx| \leqslant 1$ and $|\cos kx| \leqslant 1$, we have

$$|a_k| \leqslant \frac{1}{k^3} + 3 \cdot \frac{1}{k^2}$$

for all $x \in \mathbb{R}$. Since $\sum_{k=1}^{\infty} 1/k^3$ and $\sum_{k=1}^{\infty} 1/k^2$ are standard convergent series, it follows that $\sum_{k=1}^{\infty} |a_k|$ converges from the Comparison Test. Hence the original series converges for all x.

4. The series is telescopic. Indeed

$$\frac{1}{(2k-1)(2k+1)} = \frac{1/2}{2k-1} - \frac{1/2}{2k+1},$$

and so the series is equal to

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)(2k+1)} = \frac{1}{2}\sum_{k=1}^{\infty} \frac{1}{2k-1} - \frac{1}{2}\sum_{k=1}^{\infty} \frac{1}{2k+1} = \frac{1}{2}.$$

5. (a) We use the Integral Test. We have to evaluate the convergence of the integral

$$\int_2^{\infty} \frac{dx}{x(\ln x)^c}$$

and check that the integrand is decreasing. The last claims follows readily from the derivative of $\dfrac{1}{x(\ln x)^c}$:

$$\left(\frac{1}{x(\ln x)^c}\right)' = -\frac{1}{x^2(\ln x)^c} - \frac{c}{x^2(\ln x)^{c+1}}.$$

It is always negative, so the integrand is decreasing. For the integral, let $u = \ln x$, then

$$\int_2^{\infty} \frac{dx}{x(\ln x)^c} = \int_{\ln 2}^{\infty} \frac{du}{u^c}.$$

We know that the integral converges if and only if $c > 1$. Hence the series converges if $c > 1$ and diverges otherwise.

(b) We proceed in the same way. The derivative of the integrand is

$$\left(\frac{1}{x(\ln x)(\ln\ln x)^c}\right)' = \frac{-1}{x^2(\ln x)(\ln\ln x)^c} + \frac{-1}{x^2(\ln x)^2(\ln\ln x)^c} + \frac{-c}{x^2(\ln x)^2(\ln\ln x)^{c+1}},$$

that is negative. The convergence of the integral is measured by

$$\int_2^{\infty} \frac{dx}{x(\ln x)(\ln\ln x)^c} = \int_{\ln 2}^{\infty} \frac{du}{u(\ln u)^c} = \int_{\ln(\ln 2)}^{\infty} \frac{dv}{v^c},$$

using a second substitution $v = \ln u$. As before, the series converges if $c > 1$ and diverges otherwise.

6. Because N is finite,
$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N-1} a_n + \sum_{n=N}^{\infty} a_n.$$
Let $b_n = a_{N+n-1}$ with $g : [1, \infty) \to \mathbb{R}$ defined by $g(x) = f(N + x - 1)$. Then,
$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} f(N + n - 1) = \sum_{n=1}^{\infty} g(n)$$
with g that satisfies the hypotheses of the Integral Test and so all integrals and all series converge or diverge together.

7. As in the proof of Part 1., let $u_n = \sum_{k=1}^{n}(a_k - b_k)$. Then, $u_n = s_n - t_n$ and so
$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} s_n - \lim_{n\to\infty} t_n = a - b.$$
Finally, let
$$u_n = \sum_{k=1}^{n}(ca_k) = c(\sum_{k=1}^{n} a_k) = c \cdot s_n.$$
And so, $\lim_{n\to\infty} u_n = c(\lim_{n\to\infty} s_n) = ca$.

8. Those are geometric series. Recall that they converge if and only if their ratio r is smaller than 1 (in modulus). And so,

   (a) $r = x/3$, the series converges if $|x| < 3$, that is, $x \in (-3, 3)$ to $\dfrac{x}{3 - x}$;

   (b) $r = x - 4$, the series converges if $|x - 4| < 1$, that is, $x \in (3, 5)$ to $\dfrac{x - 4}{5 - x}$;

   (c) $r = \tan(x)$, the series converges if $|\tan(x)| < 1$, that is, $x \in (-\pi/4 + k\pi, \pi/4 + k\pi)$, $k \in \mathbb{Z}$, to $\dfrac{\tan x}{1 - \tan x} = \dfrac{\sin x}{\cos x - \sin x}$.

9. The Ratio Test take the form $\lim_{n\to\infty} \left|\dfrac{a_n + 1}{a_n}\right| = \lim_{n\to\infty} \left|\dfrac{p(n+1)}{p(n)}\right| \cdot \left|\dfrac{q(n)}{q(n+1)}\right|$. Note that the leading coefficients of $p(n + 1)$ and $p(n)$ are the same for any polynomial, and so $\lim_{n\to\infty} \left|\dfrac{p(n+1)}{p(n)}\right| = 1$, as well as for the ratio of q, therefore the Ratio Test is always inconclusive. Because the degrees of p and q are non negative integers, using the Limit Comparison Test, we see that the series converges if and only if the degree of q is larger or equal to the degree of p plus 2.

10. We know that $|a_n| \leq \rho_2^n$ for $n \geq N$. And so,
$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{N-1} |a_n| + \sum_{n=N}^{\infty} |a_n| \leq \sum_{n=1}^{N-1} |a_n| + \sum_{n=N}^{\infty} \rho_2^n \leq \sum_{n=1}^{N-1} |a_n| + \frac{\rho_2^N}{1 - \rho_2} < \infty,$$
meaning that the series converges.

Chapter 5

Deeper Results on Sequences and Series

5.1 Lecture 16: (ϵ, N)-Definition of Limits

In this lecture we give a **more general** definition of limit that does not need the existence of a function $f : [1, \infty) \to \mathbb{R}$ defining a_n. It is more precise, and somewhat simpler, than Definition 2.5, although it looks more involved.

Definition 5.1. *A sequence $\{a_n\}_{n=1}^{\infty}$ has* **limit** $\boldsymbol{a}$ *if for every $\epsilon > 0$, there is an integer $N(\epsilon)$ (usually depending on ϵ) such that, whenever $n > N(\epsilon)$, $|a_n - a| < \epsilon$. If $\{a_n\}_{n=1}^{\infty}$ has limit a we write $\lim_{n\to\infty} a_n = a$. If $\lim_{n\to\infty} a_n$ exists, we say that the sequence $\{a_n\}_{n=1}^{\infty}$* **converges**. *Otherwise we say it* **diverges**.

This definition of convergence is an extension of Definition 2.5.

Theorem 5.2. *Suppose $f : \mathbb{R} \to \mathbb{R}$ and $f(n) = a_n$ whenever $n \in \mathbb{N}$. Furthermore, suppose that $\lim_{x\to\infty} f(x) = l$. Then $\lim_{n\to\infty} a_n = l$.*

Proof. Given any $\epsilon > 0$, we know from the definition of $\lim_{x\to\infty} f(x)$ that there exists M such that, whenever $x > M$, we have $|f(x) - l| < \epsilon$. So take $N = \lceil M \rceil$, that is, M rounded **up** to the nearest integer (see Fundamentals at Level 1). Then, if $n > N$, we have $|a_n - l| = |f(n) - l| < \epsilon$ because $n > N \geq M$. □

Remark 5.3. *Because the limit of a sequence depends only on the* **(N)-tail** *of a sequence $\{x_n\}_{n=1}^{\infty}$, defined as $\{\ x_n : n > N\ \}$. Those results are valid for any tail being positive, or bounded in $[a, b]$.*

As with limit of functions, we can define what it means to have limit $\pm\infty$.

Definition 5.4. *We say that $\lim_{\mathbf{n}\to\infty} \mathbf{a_n} = +\infty$ if for every $\boldsymbol{M} \in \mathbb{R}$ there exists an integer $\boldsymbol{N}$ such that for all $\boldsymbol{n} > \boldsymbol{N}$, $\boldsymbol{a_n} \geq \boldsymbol{M}$. We say that $\lim_{\mathbf{n}\to\infty} \mathbf{a_n} = -\infty$ if for every $\boldsymbol{M} \in \mathbb{R}$ there exists an integer $\boldsymbol{N}$ such that for all $\boldsymbol{n} > \boldsymbol{N}$, $\boldsymbol{a_n} \leq \boldsymbol{M}$.*

Note that the idea in the definition is that M can be **as large as we wish**. Conversely, in the following, M is as large as we wish in modulus but **negative**.

Example. The sequence $\{2^n\}_{n=1}^{\infty}$ tends to infinity because 2^n can be made as large as we like for n large enough. Basically, for any real $M > 0$, $2^n > M$ when $n > \log_2 M$.

5.1.1 Practical Aspects of Estimates of Convergent Sequences

It is important to have a good grasp of the above definition of convergence. It implies that however small we make the interval around l, all but a finite number of terms of the sequence will lie outside of the interval. To state this in another way: the 'tail of the sequence' will lie inside the interval. The smaller we make ϵ, the larger $N = N(\epsilon)$ has to be but, when we have convergence, there is always such a value $N(\epsilon)$.

Remarks 5.5. *1. You should also note that $N(\epsilon)$ is not unique since, if we have an $N(\epsilon)$ corresponding to a given $\epsilon > 0$, then any natural number larger that this N will also suffice to satisfy the definition. This will be useful later.*

2. The definition also implies that **any number** *of the* **initial values** *of x_n* **are irrelevant** *for convergence.*

Today we shall concentrate on understanding this definition, in particular estimating the number $N(\epsilon)$. What it means is that, whatever (small) number ϵ is given, when $n > N(\epsilon)$ we know that $|x_n - l| < \epsilon$. To give an estimate to $N(\epsilon)$ we must:

1. Have a candidate for l. That candidate could come from calculating, guessing etc.
2. Find a simple **upper bound** for $|x_n - l|$ **that is simple in $\mathbb{N}$**.
3. Set the upper bound to be smaller than ϵ. If the bound is simple, then we can get easily the value of $N(\epsilon)$.

Examples.

1. A very simple example is the sequence $\{x_n\}_{n=1}^{\infty}$ with $x_n = \dfrac{1}{n}$. We believe, or calculated using our knowledge of Level 1, that $l = 0$. We can use the definition to check if this is correct. We need to evaluate (recall $n \in \mathbb{N}$)

$$|x_n - l| = |\frac{1}{n} - 0| = \frac{1}{n} < \epsilon.$$

   And so, for $n > N(\epsilon) = \lfloor 1/\epsilon \rfloor$, where $\lfloor x \rfloor$ represents the integer part of x (the largest integer smaller than x), $\dfrac{1}{n} < \epsilon$.

   Note that, as expected, $N(\epsilon)$ increases (to infinity) as ϵ tends to 0.

2. Using the sequence $\{x_n\}_{n=1}^{\infty}$ with $x_n = \dfrac{1}{n+n^2}$, we can again estimate $N(\epsilon)$. The candidate for the limit is again $l = 0$. And so,

$$|x_n - l| = \frac{1}{n+n^2} < \frac{1}{n} < \epsilon,$$

because $n^2 \geq 0$. We could use the previous value for $N(\epsilon) = \lfloor 1/\epsilon \rfloor$. This value for $N(\epsilon)$ is far from optimal as we are going to see now.

3. To get a more concrete understanding of what $N(\epsilon)$ does, consider the previous example with $x_n = \dfrac{1}{n+n^2}$. Take $\epsilon = 0.1$, say. The estimate $n \geq \dfrac{1}{\epsilon}$ is easy to calculate, basically, if $n > (0.1)^{-1} = 10$, $\dfrac{1}{n+n^2} < 0.1$, but it is not optimal. Indeed, it is enough to choose $n \geq 3$ to get $\dfrac{1}{n+n^2} \leq \dfrac{1}{12} < \dfrac{1}{10}$. Actually, in the exercises, we are going to see that, from the analysis of quadratic inequalities you did early in Level 1, if

$$n > N(\epsilon) = \left\lfloor -\frac{1}{2} + \sqrt{\frac{4+\epsilon}{4\epsilon}} \right\rfloor = \left\lfloor \frac{2}{\sqrt{\epsilon}(\sqrt{4+\epsilon}+\sqrt{\epsilon})} \right\rfloor, \tag{5.1}$$

then $\frac{1}{n+n^2} < \epsilon$. **Again, note that $N(\epsilon)$ increases as ϵ tends to 0.** But, clearly, (5.1) is very complicated, but gives the first 'exact' estimate of $N(\epsilon)$. For $\epsilon = 10^{-1}$, you get $n > [\sqrt{10.25} - 0.5] = 2$.

4. Let $x_n = 1 + \frac{\cos n}{n}$. The candidate for the limit is $l = 1$. Then,

$$|x_n - 1| = \left|\frac{\cos n}{n}\right| = \frac{|\cos n|}{n} \leq \frac{1}{n} < \epsilon,$$

and so we can choose $N(\epsilon) = 1/\epsilon$.

5. For a more complicated example, consider the sequence whose general term is

$$x_n = \frac{103n^2 - 8}{4n^2 + 99n - 3}.$$

The sequence converges because it is a combination of standard convergent sequences. We have

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{103n^2 - 8}{4n^2 + 99n - 3} = \lim_{n\to\infty} \frac{103 - 8/n^2}{4 + 99/n - 3/n^2} = \frac{103}{4}.$$

To show, using the definition, that $l = \dfrac{103}{4}$ is indeed the limit, we estimate

$$\frac{103n^2 - 8}{4n^2 + 99n - 3} - \frac{103}{4} = \frac{99 \cdot 103n - 277}{4(4n^2 + 99n - 3)} \leq \frac{100 \cdot 104n}{4(4n^2)} \leq \frac{25 \cdot 26}{n} \leq \frac{700}{n},$$

because $99n - 3 \geq 0$ for $n \geq 1$. Therefore $|x_n - l| < \epsilon$ if $n > \dfrac{700}{\epsilon} = N(\epsilon)$.

Cautionary Tale 5.6. *Note that this is highly not unique. Each of us will usually have their own expression. But, some estimates are correct, some others are not because of errors, not of choice.*

5.1.2 Divergent Sequences

Sequences that do not converge satisfy the following result.

Proposition 5.7. *A sequence $\{x_n\}_{n=1}^{\infty}$ is not convergent if and only if for all $l \in \mathbb{R}$, there exists an $\epsilon(l) > 0$ such that for all $N \in \mathbb{N}$, there exists $\bar{n}(N) > N$ such that*

$$|x_{\bar{n}} - l| \geq \epsilon(l).$$

Proof. To show that a sequence is divergent, we need to state the contraposition of the definition of a limit. For that we need to look back at some **propositional logic**. The **contraposition** of statements with the quantifiers $\exists$ or $\forall$ are as follows:

1. $\neg(\forall x,\ P(x))) = \exists x, \neg P(x)$,
2. $\neg(\exists x,\ P(x))) = \forall x, \neg P(x)$.

We use those two properties to state the contrary of the definition of a limit by exchanging $\forall$ and $\exists$. Now, in a compact notation, a sequence is convergent if

$$\exists l, \forall \epsilon, \exists N(\epsilon), \forall n > N(\epsilon), \quad |x_n - l| < \epsilon.$$

The contraposition is thus

$$\forall l, \exists \epsilon(l), \forall N, \exists \bar{n}(N), \quad |x_n - l| \geq \epsilon(l).$$

□

Example. Proposition 5.7 shows that the sequence defined by $x_n = (-1)^n$ is divergent. Indeed, let $l \in \mathbb{R}$ be any candidate for its limit. Take $\epsilon(l) = 1$. Then, for all N, either

1. $l \geq 0$, and $|x_{2N+1} - l| = |-1 - l| = 1 + l \geq 1$,
2. $l < 0$, and $|x_{2N} - l| = |1 - l| = 1 + |l| \geq 1$.

In both cases, $\bar{n}(N) = 2N + 1$, $l \geq 0$, or $2N$, $l < 0$, is such that $|x_{\bar{n}(N)} - l| \geq 1$.

Using the same type of argument we can also show the following uniqueness result showing that we can talk about **THE LIMIT** of a convergent sequence.

Theorem 5.8 (The limit is unique)**.** *If the sequence $\{x_n\}_{n=1}^{\infty}$ converges then its limit is unique.*

Proof. We use a proof by contradiction and thus we start by assuming that the converse is true. That is, we assume that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} x_n = y$ with $x \neq y$. That is we have $|x - y| > 0$. Now, by the triangle inequality, we have

$$|x - y| = |(x - x_n) + (x_n - y)| \leqslant |x_n - x| + |x_n - y|.$$

Because x and y are limits of $\{x_n\}_{n=1}^{\infty}$, for every $\epsilon > 0$ there exists $N_1(\epsilon)$ and $N_2(\epsilon)$ such that $|x_n - x| < \epsilon$, for all $n > N_1(\epsilon)$, and $|y - x_n| < \epsilon$, for all $n > N_2(\epsilon)$. We want both of these to be true and thus take $n > N(\epsilon) = \max\{N_1(\epsilon), N_2(\epsilon)\}$ and we get

$$0 < |x - y| < 2\epsilon.$$

This is true for every $\epsilon > 0$ and hence, it is true for $\epsilon = |x - y|/4$, which immediately gives us a contradiction since we cannot have

$$0 < |x - y| \leqslant \frac{2}{4}|x - y| = \frac{1}{2}|x - y|,$$

when $|x - y| \neq 0$. □

5.2 Extra curricular material: Error Estimates from the Tests

In this extra curricular material we study various error estimates of the sum of series we can deduce from the work on the tests for convergence

5.2.1 Error Estimates from the Ratio and Root Tests

We can obtain error estimates from the tests in Lemma 4.24 and Theorem 4.19.

Lemma 5.9. *Let $\{a_n\}_{n=1}^{\infty}$ be a convergent sequence of terms such that there exists an N with, either*

$$|a_{n+1}/a_n| \le \rho_1 < 1, \quad n \ge N,$$

or

$$|a_n|^{1/n} \le \rho_2 < 1, \quad n \ge N.$$

We can estimate the error between the sum s of the series and its partial sum s_n by

$$|s - s_n| \le \frac{\rho_1}{1-\rho_1}|a_n|, \quad n \ge N,$$

in the first case, or by

$$|s - s_n| \le \frac{\rho_2^{n+1}}{1-\rho_2}, \quad n \ge N,$$

in the second.

Proof. Using the notation and calculations of Lemma 4.24, we can evaluate the difference between s and s_n when $n \ge N$. We have

$$|s - s_n| \le \sum_{k=1}^{\infty} |a_{n+k}| \le \frac{\rho_1}{1-\rho_1}|a_n|.$$

For the second case, as before, comparing with the convergent geometric series we deduce that the series converges and that the error $s - s_n$ is bounded by

$$|s - s_n| \le \sum_{k=1}^{\infty} |a_{n+k}| \le \sum_{k=1}^{\infty} \rho_2^{n+k} \le \frac{\rho_2^{n+1}}{1-\rho_2}.$$

□

5.2.2 Error Estimates for the Integral Test

Theorem 5.10 (Integral Test Error Estimates). *1. If $f : [1,\infty) \to [0,\infty)$ is a decreasing continuous function with $I = \int_1^{\infty} f(x)\,dx < \infty$, so that $s = \sum_{n=1}^{\infty} f(n)$ converges. Then $s \in [I, I + f(1)]$. The 'speed' of convergence of s_n to s is evaluated by*

$$0 \leqslant s - s_n \leqslant \int_n^{\infty} f(x)\,dx. \tag{5.2}$$

2. *In the situation of Theorem 4.19, when both I and s diverge, there exists $\gamma \leq f(1)$ such that*
$$\lim_{n\to\infty}(s_{n-1} - I_n) = \gamma.$$
That is, s and I both diverge at a rate comparable with a constant less or equal to $f(1)$.

Proof. Following the proof of Theorem 4.19, we can relate the limits of $\{s_n\}_{n=1}^{\infty}$ and $\{I_n\}_{n=1}^{\infty}$ in the convergent case. We have that if $s_n \to s$ and $I_n \to I$ as $n \to \infty$ then

$$s - f(1) \leqslant I \leqslant s$$

and thus the sum of the series s satisfies $s \in [I, I + f(1)]$. We also have

$$0 \leqslant s_{n+j} - s_n = \sum_{k=n+1}^{n+j} f(k) \leqslant \int_n^{n+j} f(x)\,dx.$$

Letting $j \to \infty$ gives
$$0 \leqslant s - s_n \leqslant \int_n^{\infty} f(x)\,dx.$$
This gives a bound on how rapidly the sequence converges to s. In general, recall that

$$f(k) \leqslant \int_{k-1}^{k} f(x)\,dx \leqslant f(k-1).$$

And so,
$$0 \leqslant \left(f(k-1) - \int_{k-1}^{k} f(x)\,dx\right) \leqslant f(k-1) - f(k).$$
Summing from $k = 2$ to n gives

$$0 \leqslant s_{n-1} - I_n \leqslant f(1) - f(n) < f(1)$$

where
$$s_{n-1} - I_n = \left(f(1) - \int_1^2 f(x)\,dx\right) + \cdots + \left(f(n-1) - \int_{n-1}^{n} f(x)\,dx\right).$$
Since $\{s_{n-1} - I_n\}_{n=1}^{\infty}$ is increasing with n and bounded by $f(1)$ we know that it converges with

$$\lim_{n\to\infty}(s_{n-1} - I_n) = \gamma \leqslant f(1).$$

Thus the difference between the two sequences converges even in the case when the sequences themselves diverge. □

Examples.

1. When $f(x) = 1/x$ we obtain the result that

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n-1}\frac{1}{k} - \ln n\right) = \gamma < 1.$$

In this case it can be shown that $\gamma = 0.577..$ which is known as the **Euler**[1] **constant**. Because

$$\lim_{n\to\infty} (\ln(n+1) - \ln n) = \lim_{n\to\infty} \ln(1+\frac{1}{n}) = \ln 1 = 0,$$

we can actually write γ in a more convenient form:

$$\gamma = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \ln n \right).$$

Thus $\sum_{n=1}^{\infty} 1/n$ diverges as slowly as $\ln n$, that is, for all $\epsilon > 0$, there exists $N(\epsilon)$ such that

$$\ln(n) + \gamma - \epsilon \leq \sum_{k=1}^{n} \frac{1}{k} \leq \ln(n) + \gamma + \epsilon, \quad n \geq N(\epsilon).$$

2. In the case $f(x) = 1/x^p$, $p > 1$, we have

$$0 \leqslant s - s_n \leqslant \int_n^{\infty} \frac{dx}{x^p}.$$

Now,

$$\int_n^{\infty} \frac{dx}{x^p} = \lim_{M\to\infty} \int_n^{M} \frac{dx}{x^p} = \frac{1}{p-1} \left(\frac{1}{n^{p-1}} \right).$$

When $p = 2$, we have

$$0 \leqslant \sum_{k=n+1}^{\infty} \frac{1}{k^2} \leqslant \frac{1}{n}.$$

Hence to obtain the sum accurate to 10^{-3} we need a **thousand** (10^3) terms and to obtain the sum accurate to 10^{-6} we need a **million** (10^6) terms. The series converges rather slowly.

[1] L. Euler (1707-1783) was a Swiss mathematician who spent most of his career between St-Petersburg and Berlin. He was one of the towering lights of 18th century mathematics. He has been the most prolific mathematician of his exceptional caliber and was nearly blind for many years still producing papers on all areas of mathematics. Euler claimed that he made some of his greatest mathematical discoveries while holding a baby in his arms with other children playing round his feet (he had 13 of them, although only 5 did not die in infancy).

5.3 Lecture 17: Absolute Convergence of Series and the Leibnitz Criterion of Convergence

Coming back to Lemma 4.14, we can introduce some terminology connected with this result.

Definition 5.11. *The series $\sum_{n=1}^{\infty} a_n$ is said to be* **absolutely convergent** *if the series $\sum_{n=1}^{\infty} |a_n|$ converges. The series $\sum_{n=1}^{\infty} a_n$ is said to be* **conditionally convergent** *if $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges.*

We can thus rephrase Lemma 4.14: **absolute convergence implies convergence**. We would like here to look at what happens to a series that is not absolutely convergent.

5.3.1 Alternating Sequences

We have seen before that the fact that the sequence $\{a_n\}_{n=1}^{\infty}$ tends to 0 is not sufficient to ensure that the series $\sum_{n=1}^{\infty} a_n$ converges. Here to consider sequences when this is true when it is monotonic decreasing.

Definition 5.12. *The sequence $\{a_n\}_{n=1}^{\infty}$ is* **alternating** *if the sequence $\{(-1)^n a_n\}_{n=1}^{\infty}$ is of one sign, that is, all positive or all negative. Such sequences generate an* **alternating series** *$\sum_{n=1}^{\infty} a_n$.*

An example is the **alternating harmonic series**

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots .$$

Theorem 5.13 (Leibnitz Test). *Let $\{a_n\}_{n=1}^{\infty}$ be an alternating sequence such that $\{|a_n|\}_{n=1}^{\infty}$ is strictly monotonically decreasing with $\lim_{n\to\infty} a_n = 0$. Then the series $\sum_{n=1}^{\infty} a_n$ converges. Moreover, the rate of convergence of the partial sum s_n can be evaluated by*

$$|s - s_n| \le |a_{n+1}|, \quad n \ge 1.$$

Proof. We assume that the first term a_1 of the series is positive. When it is negative we multiply everything by (-1) and follow the same procedure. So, the series is $\sum_{n=1}^{\infty} a_n = a_1 - |a_2| + a_3 - |a_4| + \ldots$. Consider the **odd partial sums** $s_{2n-1} = \sum_{i=1}^{2n-1} a_i$ and the **even partial sums** $s_{2n} = \sum_{i=1}^{2n} a_i$. We claim that s_{2n-1} is monotone decreasing because

$$s_{2n+1} = s_{2(n+1)-1} = s_{2n-1} - |a_{2n}| + a_{2n+1} < s_{2n-1},$$

and s_{2n} is monotone increasing because

$$s_{2n+2} = s_{2(n+1)} = s_{2n} + a_{2n+1} - |a_{2n+2}| > s_{2n}.$$

Therefore we have the inequalities

$$a_1 - |a_2| = s_2 < s_{2n} < s_{2n+1} < s_1 = a_1. \tag{5.3}$$

The sequences of odd and even partial sums are monotone and bounded, hence they both converge. Their limits must be equal because

$$\lim_{n\to\infty}(s_{2n+1}-s_{2n}) = \lim_{n\to\infty} a_{2n+1} = 0.$$

Therefore $s = \lim_{n\to\infty} s_{2n+1} = \lim_{n\to\infty} s_{2n} = \lim_{n\to\infty} s_n$.

The relation (5.3) means that the partial sums of an alternating series also alternates above and below the final limit s. More precisely, when there are odd (even) number of terms, i.e. the last term is a plus (minus) term, then the partial sum is above (below) the final limit. This leads immediately to the error bound of partial sums, shown below. We have

$$\begin{aligned} |s_{2n+1}-s| &\leq s_{2n+1}-s \leq s_{2n+1}-s_{2n+2} = |a_{2n+2}|, \\ |s_{2n}-s| &\leq s-s_{2n} \leq s_{2n+1}-s_{2n} = a_{2n+1}. \end{aligned}$$

And so we can summarise both inequalities as $|s-s_n| \leq |a_{n+1}|$, $n \geq 1$. □

Example. The Leibnitz result can be useful to study the behaviour of power series at the of convergence because we get an alternating series at $x = -R$.

A series is determined by the sequence of its partial sums. So the order in which we sum its terms is in principle important. Absolutely convergent series have the remarkable property that their sum is independent of the order of summation, hence their name.

Definition 5.14. *A series $\sum_{n=1}^{\infty} b_n$ is a* **re-arrangement** *of a series $\sum_{n=1}^{\infty} a_n$ if there exists a bijection $\phi : \mathbb{N} \to \mathbb{N}$ such that $b_n = a_{\phi(n)}$.*

Theorem 5.15 (Re-arrangement Theorem)**.** *Every re-arrangement of an absolutely convergent series is absolutely convergent and all have the same sum.*

Proof. We proceed in two steps. First we show that the result is true for the re-arrangement of a series with positive terms. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence with positive terms and $\{b_n\}_{n=1}^{\infty}$ be a re-arrangement given by a bijection $\phi : \mathbb{N} \to \mathbb{N}$. Let $s_n = \sum_{k=1}^{n} a_k$ and $t_n = \sum_{k=1}^{n} b_k$. For each n, choose $m(n) \geq \max(\phi(1), \cdots, \phi(n))$. Then $t_{m(n)} \geq s_n$ because all the terms in s_n appear in $t_{m(n)}$ If the series $\sum_{n=1}^{\infty} b_n$ exists,

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n \leq \lim_{n\to\infty} t_{m(n)} = \sum_{n=1}^{\infty} b_n.$$

Note that $\{a_n\}_{n=1}^{\infty}$ is also a re-arrangement of $\{b_n\}_{n=1}^{\infty}$ via ϕ^{-1}. And so, both sums exist and are equal because

$$\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n \leq \sum_{n=1}^{\infty} a_n.$$

In a second part we consider an absolutely convergent series $s = \sum_{n=1}^{\infty} a_n$ and a re-arrangement $t = \sum_{n=1}^{\infty} b_n$. Let $s_n^+ = \sum_{k=1,\, a_k>0}^{n} a_k$ and $s_n^- = \sum_{k=1,\, a_k<0}^{n} (-a_k)$ be the (positive) partial sums of the series of positive, respectively negative, terms of the original series. Similarly, let $\{t_n^+\}_{n=1}^{\infty}$ and $\{t_n^-\}_{n=1}^{\infty}$ be the (positive) partial sums of the series of positive, respectively negative, terms of the re-arranged series. Because the original series is absolutely convergent, s_n^+ and s_n^- converge and have positive terms. It is clear that any re-arrangement of the sequence $\{a_n\}_{n=1}^{\infty}$ does not alter the sign of its terms and so induces re-arrangements of the sequences of positive, respectively, negative, terms. From the first part we know that $\{s_n^+\}_{n=1}^{\infty}$ and $\{t_n^+\}_{n=1}^{\infty}$, respectively $\{s_n^-\}_{n=1}^{\infty}$ and $\{t_n^-\}_{n=1}^{\infty}$, converge to the same limit. And so the conclusion because

$$s = \lim_{n\to\infty} s_n = \lim_{n\to\infty}(s_n^+ - s_n^-) = \lim_{n\to\infty}(t_n^+ - t_n^-) = \lim_{t\to\infty} t_n = t.$$

□

Conditionally convergent sequences have a remarkable property: they are formed of entwined divergent series of positive, resp. negative, terms.

Lemma 5.16. *If a series $\sum_{n=1}^{\infty} a_n$ is conditionally convergent, then the series of its positive, respectively negative terms, are divergent.*

Proof. Let $s_n^+ = \sum_{k=1,\, a_k>0}^{n} a_k$ and $s_n^- = \sum_{k=1,\, a_k<0}^{n}(-a_k)$ be the (positive) partial sums of the series of positive, respectively negative, terms of $\sum_{n=1}^{\infty} a_k$. Recall that $s_n = s_n^+ - s_n^-$.We are going to show that if one of the sequences $\{s_n^+\}_{n=1}^{\infty}$ or $\{s_n^-\}_{n=1}^{\infty}$ converges then the other must also converge, hence the original series must have been absolutely converging. Suppose that $\lim_{n\to\infty} s_n^+ = s^+ < \infty$. Define $s^- = s^+ - s$ where s is the sum of the original series. We evaluate

$$|s^- - s_n^-| \leq |s^+ - s - s_n^-| \leq |-s + s_n^+ - s_n^-| + |s^+ - s_n^+| = |s - s_n| + |s^+ - s_n^+|.$$

From the convergence of the sequences $\{s_n\}_{n=1}^{\infty}$ and $\{s_n^+\}_{n=1}^{\infty}$ we can conclude that $\{s_n^-\}_{n=1}^{\infty}$ is also convergent, a contradiction with the fact that the original series was not absolutely convergent. □

Finally, a remarkable result dues to Riemann[2].

Theorem 5.17. *If a series $\sum_{n=1}^{\infty} a_n$ converges conditionally, then for any number $s \in \mathbb{R}$ there exists a re-arrangement $\sum_{n=1}^{\infty} b_n$ converging to s.*

Proof. The basis of the proof is that the partial sums of the positive and negative numbers must both diverge as was shown in Lemma 5.16. Then we use the trick shown in the lectures to show the result for the alternating harmonic series. Suppose $c \geq 0$ (the proof follows exactly the same line if $c < 0$). We take positive terms of the series until we go over c for the first time. Then we take negative terms to get for the first time under c. Then we retake positive terms to go over c again, etc. At each step we go over or under c by a strictly smaller amount, hence the final convergence to c of the re-arrangement of the series. □

[2]G.F.D. Riemann (1826-1866) was a German mathematician of great originality. He developed complex analysis and the relation between geometry and analysis. In his exceptional PhD thesis, he already showed his style: great insight but somewhat lacking of the necessary rigour. On the other hand, the work of other mathematicians to fill in the gaps in his ideas proved very fertile in the development of mathematics. One of the very important conjectures in mathematics is about the structure of the complex zeros of the Riemann's ζ function.

Example

Conditionally convergent series are difficult objects to manipulate. For instance, let

$$
\begin{aligned}
s &= \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i} = \sum_{i=1}^{\infty} \left(\frac{1}{2i-1} - \frac{1}{2(2i-1)} - \frac{1}{4i} \right) \text{(by re-arrangement)} \\
&= \sum_{i=1}^{\infty} \left(\frac{1}{2(2i-1)} - \frac{1}{4i} \right) = \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{2i} = \frac{1}{2} \left(\sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i} \right) = \frac{1}{2}\, s.
\end{aligned}
$$

Hence $s = 0$. But $s > 0$ (in Exercise 5 of Sheet 4a we show that it is $\ln 2$). This contradiction occurs because we have kept the equal sign with the re-ordering of the series which is not possible unless the series are absolutely convergent.

Summary of Chapter 4

We have seen:

- The (ϵ, N)-definition of a limit and to be able to evaluate $N(\epsilon)$ in simple cases;
- To know how to apply the Ratio, Root, Integral and Leibniz Tests for convergence of series.
- **further tests** for determining whether a series is convergent;
- that we must **check carefully** that it is appropriate to use either of the tests;
- To know the definitions and to understand the difference between absolute and conditional convergence of series.

5.4 Exercise Sheet 4

5.4.1 Exercise Sheet 4a

1. You have already calculated most of the following limits in Sheet 2a. Show using the (ϵ, N)-definition that those limits are actually correct. To be precise,

   (a) Determine, when it exists, a 'candidate' for the limit l for the following sequences $\{x_n\}_{n=1}^{\infty}$.

   (b) When such 'candidate' limit l exists. Determine an $N(\epsilon)$ such that the definition of the limit of a sequence gives you that $|l - x_n| < \epsilon$, for all $n > N(\epsilon)$. Recall that $N(\epsilon)$ does not need to be the smallest value (although you might try to have fun to get a small one).

   (c) When such 'candidate' limit l does not exists, show that the sequence has no limit using the definition (more precisely its converse).

   (a) $x_n = -n + \sqrt{n^2 + 3n}$. (b) $x_n = \dfrac{5n^3 + 3n + 1}{15n^3 + n^2 + 2}$. (c) $x_n = \dfrac{\sin(n^2 + 1)}{n^2 + 1}$.
   (d) $x_n = \dfrac{\sqrt{n+2} - \sqrt{n+1}}{\sqrt{n+1} - \sqrt{n}}$. (e) $x_n = \dfrac{7n^4 + n^2 - 2}{14n^4 + 5n - 4}$. (f) $x_n = \dfrac{n^3 + 3n^2}{n+1} - n^2$.
   (g) $x_n = \left(\dfrac{n+1}{n}\right)^3 - n^3$.

2. (a) Consider the sequence $\{x_n\}_{n=1}^{\infty}$ with $x_n = \dfrac{1}{n + n^2}$. We would like to estimate the **exact value** of $N(\epsilon)$ such that

$$|x_n - l| = \frac{1}{n + n^2} < \epsilon. \tag{5.4}$$

   Note that rewriting (5.4), we would like to find $N(\epsilon)$ such that, if $n > N(\epsilon)$,

$$n^2 + n - \frac{1}{\epsilon} > 0.$$

   Use the analysis of quadratic inequalities you did early in Level 1, to find that

$$N(\epsilon) = \left\lfloor -\frac{1}{2} + \sqrt{\frac{4+\epsilon}{4\epsilon}} \right\rfloor = \left\lfloor \frac{2}{\sqrt{\epsilon}(\sqrt{4+\epsilon} + \sqrt{\epsilon})} \right\rfloor.$$

   (b) Show that $N(\epsilon)$ increases as ϵ tends to 0.

3. Prove the Limit Comparison Theorem, Theorem 4.18.

4. Let $0 < a < 1$. Consider the series

$$a^2 + a + a^4 + a^3 + \cdots + a^{2n} + a^{2n+1} + \cdots .$$

   Show that the Root Test applies but not the Ratio Test.

5. * Let $c_n = \left(\sum_{k=1}^{n} \frac{1}{k}\right) - \ln n$. Show that (c_n) is a decreasing sequence of positive numbers. Let γ be its limit. Show that if $b_n = \sum_{k=1}^{2n}(-1)^{k+1}\frac{1}{k}$, then $\lim\limits_{n\to\infty} b_n = \ln 2$.

6. Using the (ϵ, N)-definition of a limit, prove that

   (a) a convergent sequence is bounded;

   (b) If $\{x_n\}_{n=1}^{\infty}$ is a convergent sequence of real numbers and if $x_n \geqslant 0$ for all $n \in \mathbb{N}$ then
   $$l = \lim_{n\to\infty} x_n \geqslant 0.$$
   Similarly, if $x_n \leq 0$, then $l \leq 0$.

7. Using the (ϵ, N)-definition of a limit, prove that the algebra of limit holds. Namely, let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be convergent sequences in $\mathbb{R}$ with limits x and y, respectively, and let $\alpha \in \mathbb{R}$. Then,

   (a) $\lim_{n\to\infty}(x_n + y_n) = x + y$,

   (b) $\lim_{n\to\infty}(\alpha x_n) = \alpha x$,

   (c) $\lim_{n\to\infty}(x_n y_n) = xy$,

   (d) $\lim_{n\to\infty}(x_n/y_n) = x/y$ provided $y_n \neq 0$ and $y \neq 0$;

8. Suppose that $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ are sequences of real numbers such that
   $$x_n \leqslant y_n \leqslant z_n,$$
   for all $n \in \mathbb{N}$, and that
   $$\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n.$$
   Using the (ϵ, N)-definition of a limit, show that the 'intermediate' sequence $\{y_n\}_{n=1}^{\infty}$ is convergent and
   $$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n.$$

5.4.2 Additional Exercise Sheet 4b

1. When the following limits exist, determine an $N(\epsilon)$ satisfying the (ϵ, N)-definition of limit. (a) $x_n = \dfrac{n^3+2n^2+1}{6n^3+n+4}$. (b) $x_n = \dfrac{n^2+n+1}{3n^2+4}$. (c) $x_n = \sqrt{n^4+n^2} - n^2$.
   (d) $x_n = \left(\dfrac{n+1}{n^2}\right)^4 - n^4$. (e) $x_n = -n + \sqrt{n^2+n}$. (f) $x_n = \dfrac{\sin n}{n} + (\sqrt{n+1} - \sqrt{n})$.
   (g) $x_n = \dfrac{n^2+500n+1}{5n^2+3}$. (h) $x_n = \sqrt{n(n+1)} - \sqrt{n(n-1)}$.

5.4.3 Short Feedback for Exercise Sheet 4a

1. (a) $l = 3/2$,

   (b) $l = 1/3$,

   (c) $l = 0$,

(d) $l = 1$,
(e) $l = 1/2$,
(f) $l = \infty$ (divergent),
(g) $l = -\infty$ (divergent).

2. (a) Consider $1/\epsilon$ as a parameter perturbing the parabola $y(x) = x^2 + x$.
(b) Show that the derivative of the denominator is monotone.

3. Show that the limit condition means that for any ϵ there exists $N(\epsilon)$ such that

$$(l - \epsilon)\, b_n < a_n < (l + \epsilon)\, b_n, \quad \forall n > N(\epsilon).$$

Then use that relation to show that the tails of the series $\displaystyle\sum_{n=N(\epsilon)}^{\infty} a_n$ and $\displaystyle\sum_{n=N(\epsilon)}^{\infty} b_n$ converge or diverge together.

4. The coefficient ρ_2 in the Root Test has a limit, but ρ in the Ratio Test has not.

5.4.4 Short Feedback for the Additional Exercise Sheet 4b

1. (a) $l = 1/6$,
(b) $l = 1/3$,
(c) $l = 1/2$,
(d) $l = -\infty$, (divergent),
(e) $l = 1/2$,
(f) $l = 0$,
(g) $l = 1/5$,
(h) $l = 1/2$.

This result follows from the following more sophisticated test, but whose proof must be delayed until next term. Note that our results are valid if the tail of the series start to be alternating for $n \geq N$. Those series form a good source of conditionally convergent series if they are not absolutely convergent. The Leibnitz Test is the consequence of the more sophisticated Dirichlet Test.

Abel[3] Test.

Proposition 5.18 (Dirichlet Test). *Let $\{b_n\}_{n=1}^{\infty}$ be a decreasing sequence such that $\lim_{n\to\infty} b_n = 0$ and $\{a_n\}_{n=1}^{\infty}$ a sequence such that $m \leq \sum_{i=1}^{n} a_i \leq M$ for all n. Then $\sum_{k}^{\infty} a_k b_k$ converges.*

Proof. See MA2731, Analysis 2. □

[3] N.H. Abel (1802-1829) was a brilliant Norvegian mathematician that died young of illness (mainly due to poverty). He was the first to prove that there is no general formula for the solution of a quintic equation.

5.4.5 Feedback for Exercise Sheet 4a

Recall that for positive numbers

$$\frac{a}{b} \leq \frac{a+p}{b}, \quad \frac{a}{b+p} \leq \frac{a}{b}$$

and $a - b = \dfrac{a^2 - b^2}{a+b}$.

1. In that exercise, calculate the limit, when it exists, using the usual tools. Then, estimate $|x_n - l|$ trying to get a simple expression that obviously tends to 0 as n is big. Then we can find $N(\epsilon)$ by estimating how big n should be for the estimate of $|x_n - l|$ to be smaller than ϵ. **Note that this is highly not unique. Everybody will usually have their own expression, but some estimates are correct, some others are not, this is the difference between being correct or wrong**.

   (a) The limit can be calculated multiplying by the conjugate quantity

   $$\lim_{n\to\infty} x_n = \lim_{n\to\infty} (-n + \sqrt{n^2+3n}) = \lim_{n\to\infty} \frac{3n}{n + \sqrt{n^2+3n}} = \frac{3}{2}.$$

   The estimate

   $$\begin{aligned} |x_n - l| &= \left| \frac{3}{2} - \frac{3n}{n+\sqrt{n^2+3n}} \right| = \frac{3}{2} \cdot \frac{\sqrt{n^2+3n} - n}{n + \sqrt{n^2+3n}} \\ &\leq \frac{3}{2} \cdot \frac{3n}{(n+\sqrt{n^2+3n})^2} \leq \frac{9n}{2(2n)^2} = \frac{9}{8n} \leq \frac{2}{n}. \end{aligned}$$

   Clearly, $|x_n - l| < \epsilon$ when $n > \dfrac{2}{\epsilon}$.

   (b) As in (i), we show that $\lim_{n\to\infty} x_n = \dfrac{1}{3}$. To find an estimate for $N(\epsilon)$,

   $$|x_n - l| = \frac{|-n^2+9n+1|}{3(15n^3+n^2+2)} \leq \frac{|n^2-9n-1|}{3(15n^3)} \leq \frac{1}{45n},$$

   because $|n^2 - 9n - 1| \geq 1$ when $n \geq 3$. And so, $|x_n - l| < \epsilon$ when $n > \dfrac{1}{45\epsilon}$.

   (c) Here, we can estimate directly

   $$|x_n| \leq \left| \frac{\sin(n^2+1)}{n^2+1} \right| \leq \frac{1}{n^2}.$$

   This provides that $\lim_{n\to\infty} x_n = 0$ and an estimate for $N(\epsilon) = \dfrac{1}{\sqrt{\epsilon}}$.

   (d) Applying the identity $a - b = (a^2 - b^2)/(a+b)$ to the numerator and the denominator gives

   $$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+2} + \sqrt{n+1}} = 1.$$

We then estimate

$$\begin{aligned}|x_n - 1| &= \frac{|\sqrt{n+2} - \sqrt{n}|}{|\sqrt{n+2} + \sqrt{n+1}|} \\ &= \frac{2}{|\sqrt{n+2} + \sqrt{n}| \cdot |\sqrt{n+2} + \sqrt{n+1}|} = \frac{2}{(2\sqrt{n})^2} = \frac{1}{2n},\end{aligned}$$

and so, a possibility for $N(\epsilon) = \dfrac{1}{2\epsilon}$.

(e) Clearly $\lim_{n\to\infty} x_n = \frac{1}{2}$. And so, when $n \geq 2$,

$$|x_n - l| = \left|\frac{2n^2 - 5n}{2(14n^4 + 15n - 4)}\right| = \frac{2n^2}{28n^4} \leq \frac{1}{14n^2} \leq \frac{1}{9n^2},$$

and so we can choose $N(\epsilon) = \dfrac{1}{3\sqrt{\epsilon}}$.

(f) Simplifying x_n, we find that $x_n = \dfrac{2n^2}{n+1}$. Clearly then, $x_n = \dfrac{2n^2}{n+1} \geq \dfrac{2n^2}{2n} = n$, and so the sequence diverges as $x_n > M$ if $n > M$.

(g) The term $\left(\dfrac{n+1}{n}\right)^3$ is bounded below by 1 and above by 8. This follows from

$$1 \leq \frac{n+1}{n} \leq 1 + \frac{1}{n} \leq 2.$$

And so x_n is unbounded, with $\lim_{n\to\infty} x_n = -\infty$. Given an arbitrary positive number M, we can estimate for which n is $x_n < -M$. We have

$$x_n \leq 8 - n^3 < -M$$

when $n > \sqrt[3]{8 + M}$.

2.

3.

4. The Ratio Test gives $\dfrac{a_{n+1}}{a_n} = a^3 < 1$, if n is even, or $a^{-1} > 1$, if n is odd. So we cannot conclude.

On the other hand, note that if n is odd, $a_n = a^{n+1}$, and if n is even, $a_n = a^{n-1}$. The Root Test gives $(a_n)^{1/n} = a^{1+1/n}$ if n is even and $a^{1-1/n}$ if n is odd. So the limit of the root $(a_n)^{1/n}$ as n tend to ∞ is $a < 1$ in both cases as $\lim\limits_{n\to\infty} \dfrac{1}{n} = 0$. Thus the series converges.

5. * That $c_n > 0$ follows from the estimates in Theorem 4.19 and Example 2 after it. That it is decreasing follows from the following estimate:

$$c_{n+1} - c_n = \frac{1}{n+1} - \ln(n+1) + \ln n = \frac{1}{n+1} + \ln\left(1 - \frac{1}{n+1}\right).$$

Let $x = \dfrac{1}{n+1}$. So as n tends to ∞, x tends to 0^+. Now, in terms of x,

$$c_{n+1} - c_n = x + \ln(1-x) = \ln(e^x(1-x)).$$

The function $f(x) = e^x(1-x)$ is strictly smaller than 1 when x tends to 0^+ because $f(0) = 1$ and $f'(x) = -e^x < 0$. And so $c_{n+1} - c_n = \ln(f(x)) < 0$. Therefore $\{c_n\}_{n=1}^{\infty}$ tends to a limit which we called γ, the Euler constant. Note that

$$b_n = c_{2n} - c_n + \ln 2$$

because

$$\begin{aligned}
c_{2n} - c_n + \ln 2 &= \left(\sum_{k=1}^{2n} \frac{1}{k} - \ln(2n)\right) - \left(\sum_{k=1}^{n} \frac{1}{k} - \ln n\right) + \ln 2 \\
&= \sum_{k=1}^{2n} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k} - \ln 2 - \ln n + \ln n + \ln 2 \\
&= \sum_{k=1}^{2n} \frac{1}{k} - 2\left(\sum_{k=1}^{n} \frac{1}{2k}\right) = \sum_{k=1}^{2n} \frac{1}{k} - 2\left(\sum_{j=1,\text{even}}^{2n} \frac{1}{j}\right) \\
&= \sum_{k=1}^{2n} (-1)^{k+1} \frac{1}{k} \\
&= b_n.
\end{aligned}$$

Hence,

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} (c_{2n} - c_n + \ln 2) = \gamma - \gamma + \ln 2 = \ln 2.$$

6. (a)

   (b) We use a proof by contradiction. That is, we start with the assumption that $l < 0$. But the elements of the tail of the sequence can be arbitrarily close to l and in particular this implies that for sufficiently large n, $x_n < 0$. In terms of ϵ's, we take $\epsilon = -x/2 = |l|/2 > 0$. The convergence means that there exists an $N \in \mathbb{N}$ such that $|x_n - l| < \epsilon$, that is, $l < x_n - l < -l$ or $2l < x_n < 0$ for all $n > N$. This contradicts $x_n \geqslant 0$ for all $n \in \mathbb{N}$.

7. We will only prove the first and third assertions. The others are proved in the same spirit.

   Let $N_1(\epsilon)$, $N_2(\epsilon)$, be such that $|x_n - x| < \epsilon$, $|y_n - y| < \epsilon$, for $n > N_1(\epsilon)$, $n > N_2(\epsilon)$, respectively. For the first assertion we evaluate

$$|(x+y) - (x_n + y_n)| \leq |x - x_n| + |y - y_n| < \epsilon$$

   for $n > \max\{N_1(\frac{\epsilon}{2}), N_2(\frac{\epsilon}{2})\}$.

   For the third assertion, the key to the proof is to re-write the difference $x_n y_n - xy$ in a form where we can compare x_n with x and y_n with y. We have

$$x_n y_n - xy = x_n(y_n - y) + y(x_n - x).$$

Then by considering the absolute value, and using the triangle inequality, we have

$$|x_n y_n - xy| \leqslant |x_n| \cdot |y_n - x| + |y| \cdot |x_n - x|.$$

Next we use a previous result that $\{x_n\}_{n=1}^{\infty}$ being convergent implies that $|x_n|$ is bounded by a number $M > 0$. Hence

$$\begin{aligned} |x_n y_n - xy| &\leqslant |x_n| \cdot |y_n - y| + |y| \cdot |x_n - x| \\ &\leqslant M \cdot |y_n - y| + |y| \cdot |x_n - x|. \end{aligned}$$

Now take $n > \max\{N_1(\frac{\epsilon}{2|y|}), N_2(\frac{\epsilon}{2M})\}$, then

$$|x_n y_n - xy| < \epsilon.$$

8. This is a direct proof. Let
$$w = \lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n.$$
The convergence of $\{x_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ to w implies that given any $\epsilon > 0$ there exists $N(\epsilon) \in \mathbb{N}$ (the maximum of the N's for the two sequences) such that

$$|x_n - w| < \epsilon \quad \text{and} \quad |z_n - w| < \epsilon \quad \text{for all } n > N(\epsilon).$$

Now the ordering of the elements $x_n \leqslant y_n \leqslant z_n$ implies

$$x_n - w \leqslant y_n - w \leqslant z_n - w$$

and hence, for $n \geqslant N(\epsilon)$,

$$-\epsilon < x_n - w \leqslant y_n - w \leqslant z_n - w < \epsilon,$$

and so $|y_n - w| < \epsilon$. Thus $\lim_{n\to\infty} y_n = w$.

Chapter 6

Approximation with Taylor Polynomials

6.1 Lecture 18: Taylor's theorem and error estimates

It is now time to put all of these results to work. In Lectures 18 and 19 we begin to draw all of our results together and return to Taylor and Maclaurin poynomials. Our fundamental question, which we have dodged until now, is:

When is it true that the degree of the Taylor or Maclaurin polynomial can be taken to the limit and so produce legitimate Taylor or Maclaurin series of the form,

$$f(x) = \sum_{n=0}^{\infty} \frac{(x-a)^n}{n!} f^{(n)}(a)?$$

This chapter and the next diverge somewhat from the content of Lectures 18 and 19 by providing an extensive set of worked examples. Our strategy is to return to the remainder term for Taylor and Maclaurin polynomials that was introduced much earlier in Section 5. We derive the precise form of this remainder and are then able to use it for two primary purposes.

- By terminating the Taylor or Maclaurin polynomial at some finite degree, we can estimate the size of the remainder to determine a bound on the accuracy with which the polynomial, $T_n^a f(x)$ approximates $f(x)$ at x. By inverting the question, we can then determine how many terms we need to reach a specified accuracy.
- By examining the limiting behaviour of the remainder as the polynomial degree tends to infinity we can ascertain whether or not $f(x) = \lim_{n\to\infty} T_n^a f(x)$. If this limit holds then we will have rigorously proven that $f(x) = T_\infty^a f(x)$.

The second item is the focus of Lectures 18 and 19. It is without a doubt harder to grasp and so is better presented 'live'. Once the structure of the remainder term is understood though, the first item can provide ample opportunity for investigation. The first item is therefore supported mainly by this document.

Section 6.2 discusses the remainder term and then in Section 6.3 we show how this remainder can be estimated.

Concrete examples on estimating the error in replacing some familiar functions by their Maclaurin series are then detailed in Subsection 6.3.1, for computing e; Subsection 6.3.2, for the error in $T_4 \cos(x)$; and, Subsection 6.3.3, for $\sin(x)/x$.

Furthermore, Section 6.4 discusses the possibility of approximating integrals by replacing integrands with their Taylor or Maclaurin polynomials.

Once again, though, note that the main points will be covered in the lectures and detailed on the lecture slides. This document supports that activity but does not replace it.

There is one important lemma that we will make use of in the lectures. The proof is boardwork for Lecture 19.

Lemma 6.1.

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0 \qquad \forall x \in \mathbb{R}.$$

6.2 Taylor Theorem

Given a n-times differentiable function $f : I \to \mathbb{R}$, where I is an open interval and $a \in I$, we have seen that its Taylor polynomial of degree n at a

$$T_n^a f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n. \tag{6.1}$$

This notation encodes the data we need to know: the function f, the degree n and the point a. We had seen that the function f and its Taylor polynomial $T_n^a f$ share the same values at a as well as their first n derivatives. What about in an interval around $x = a$? Many mathematical operations need the values of a function on an interval, not only at one given point $x = a$. Consider an interval I around a, a tolerance $\epsilon > 0$ and the order n of the approximation, here are the big questions:

1. Given $T_n^a f$ and I. **Within what tolerance** does $T_n^a f$ approximate f on I?
2. Given $T_n^a f$ and ϵ. **On how large an interval** I does $T_n^a f$ achieve that tolerance?
3. Given f, $a \in I$ and ϵ. Find **how many terms** n must be used for $T_n^a f$ to approximate f to within ϵ on I.

Having a polynomial approximation that works **all along an interval** is a much more substantive property than evaluation at a single point. The Taylor polynomial $T_n^a f(x)$ is almost never exactly equal to $f(x)$, but often it is a good approximation, especially if $|x-a|$ is small.

Remark 6.2. *It must be noted that there are also* **other ways** *to approach the issue of* **best approximation by a polynomial on an interval**. *And beyond worry over approximating the values of the function, we might also want the values of one or more of the derivatives to be close, as well. The theory of* **splines** *is one approach to approximation which is very important in practical applications. Also the* **approximation in the mean** *can be useful: given a function f, which polynomial p of degree n minimises the error*

$$\int_a^b (f(x) - p(x))^2 dx?$$

To see how good is the approximation $T_n^a f$ of f, we defined the 'error term' or, 'remainder term'.

Definition 6.3. *Let $I \subset \mathbb{R}$ be an open interval, $a \in I$ and $f : I \to \mathbb{R}$ be a n-times differentiable function, then*

$$\boldsymbol{R_n^a f(x)} = f(x) - T_n^a f(x) \tag{6.2}$$

is called the **n-th order remainder (or error) term** *of the Taylor polynomial $T_n^a f$ of f.*

In general, we cannot really do much with Definition 6.3 on its own.

Example. If $f(x) = \sin x$ then we have found that $T_3 f(x) = x - \frac{1}{6}x^3$, so that

$$R_3 \sin(x) = \sin x - x + \tfrac{1}{6}x^3.$$

Although correct, it is not a very useful formula as it stands.

But, there is an important example where we can (**un-usually**) compute $R_n f(x)$.

Example. Consider the function $f : (-1, 1) \to \mathbb{R}$ given by

$$f(x) = \frac{1}{1-x}.$$

We have seen that the McLaurin polynomials (at $a = 0$) of f is

$$T_n f(x) = \sum_{k=0}^{n} x^k = \frac{1 - x^{n+1}}{1-x} = f(x) - \frac{x^{n+1}}{1-x},$$

a **geometric progression**. The remainder term therefore is

$$R_n f(x) = f(x) - T_n f(x) = \frac{x^{n+1}}{1-x}.$$

Our main theorem is as follows. It gives a formula for the remainder, formula we will be able to use for estimates later.

Theorem 6.4 (Taylor's Theorem)**.** *Let f be an $n+1$ times differentiable function on some open interval I containing a. Then, for every x in the interval I there is a ξ between a and x such that*

$$R_n^a f(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}. \tag{6.3}$$

Proof. Suppose $a < x$, otherwise interchange them. The full proof is not significantly harder, but we are going to show the previous result for $n = 1$ only, that is,

$$R_2^a f(x) = \frac{f''(\xi)}{2!}(x-a)^2,$$

for some $a < \xi < x$. Define the function in t, $F : [a, x] \to \mathbb{R}$ by

$$F(t) = f(x) - f(t) - f'(t)(x - t).$$

Then $F'(t) = -f''(t)(x - t)$. Define $G : [a, x] \to \mathbb{R}$ by

$$G(t) = F(t) - \left(\frac{x - t}{x - a}\right)^2 F(a).$$

Then $G(a) = G(x) = 0$. Applying Rolle's Theorem, there exists $a < \xi < x$ such that

$$0 = G'(\xi) = F'(\xi) + 2F(a)\,\frac{(x - \xi)}{(x - a)^2}.$$

Hence,

$$F(a) = -\frac{(x - a)^2}{2(x - \xi)} F'(\xi) = \frac{f''(\xi)}{2}\,(x - a)^2.$$

□

The formula (6.3) is called the **Lagrange Remainder Formula**. Even though you usually cannot compute the mystery point ξ precisely, Lagrange's formula allows you to **estimate** it. A couple of example of a type already done.

Example. Approximate $\sqrt{x}$ by a Taylor polynomial of degree 2 at $x = 4$. Use this polynomial to approximate $\sqrt{4.1}$ and use Taylor's Theorem to estimate the error in this approximation.

Let $f : (0, \infty) \to \mathbb{R}$, $f(x) = \sqrt{x}$. Then $f'(x) = \dfrac{1}{2\sqrt{x}}$, $f''(x) = -\dfrac{1}{4x^{3/2}}$ and $f'''(x) = \dfrac{3}{8x^{5/2}}$. The Taylor polynomial, $T_2^4 f(x)$, is

$$T_2^4 f(x) = f(4) + f'(4)(x - 4) + \frac{1}{2} f''(4)(x - 4)^2 = 2 + \frac{1}{4}(x - 4) - \frac{1}{64}(x - 4)^2.$$

On substituting $x = 4.1$,

$$\sqrt{4.1} \approx 2 + \frac{1}{4}(0.1) - \frac{1}{64}(0.1)^2 = 2.02484375.$$

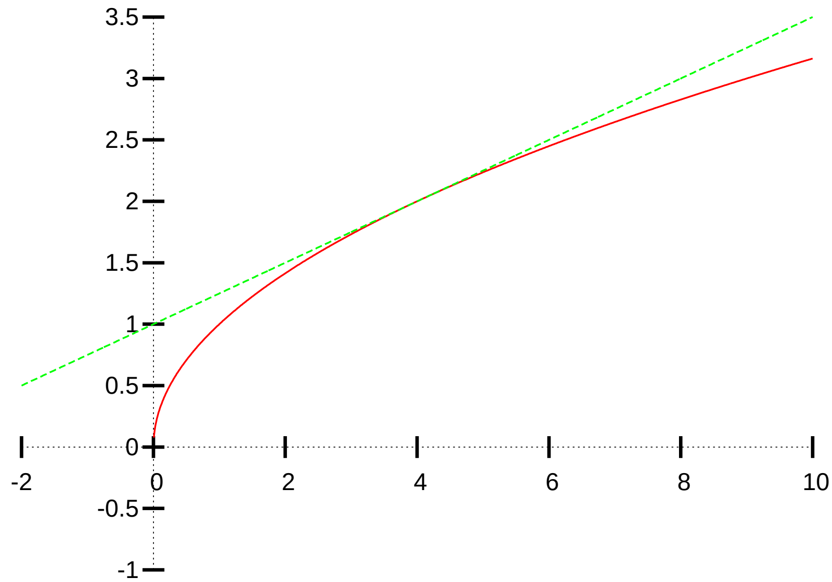
Taylor's Theorem gives

$$|R_2^4 f(4.1)| \le \frac{M(4.1 - 4)^3}{3!},$$

where M is the absolute maximum of $|f^{(3)}(x)|$ on $[4, 4.1]$. Now $f^{(3)}(x) = \dfrac{3}{8}x^{-5/2}$. This is a decreasing positive function so the absolute maximum of $|f^{(3)}(x)| = \dfrac{3}{8}x^{-5/2}$ occurs when $x = 4$. Hence, $M = \dfrac{3}{8 \cdot 4^{5/2}} = \dfrac{3}{256}$ and it follows that

$$|R_2^4 f(4.1)| \le \frac{(3/256)(0.1)^3}{3!} \approx 0.000002.$$

Thus, our approximation to $\sqrt{4.1}$ differs from the exact value by 0.000002 or less!



The graph shows $\sqrt{x}$ **in red** and $T_2^4 f$ **in green** and shows clearly that the Taylor polynomial $T_2^4 f$ approximates $\sqrt{x}$ extremely well but **only** for a certain range of x.

6.3 Estimates Using Taylor Polynomial

Here is the most common way to estimate the remainder.

Theorem 6.5 (Estimate of the Remainder Term)**.** *Let f be an $n+1$ times differentiable function on some open interval I containing a. If you have a constant M such that*

$$\left|f^{(n+1)}(\xi)\right| \leq M, \tag{6.4}$$

for all ξ between a and x, then

$$|R_n^a f(x)| \leq \frac{M|x-a|^{n+1}}{(n+1)!}. \tag{6.5}$$

Proof. We do not know what ξ is in Lagrange's formula, but it doesn't matter, for wherever it is, it must lie between a and x so that our assumption (6.4) implies that $|f^{(n+1)}(\xi)| \leq M$. Put that in Lagrange's formula and you get the stated inequality. □

Some final remark before we embark in using those results.

1. Note, usually we will find M by finding the absolute max and min of $f^{(n+1)}(x)$ on the interval I. Sometimes, however, we can find a value for M without calculating absolute max's and min's. For example, if $f(x)$ equals $\sin(x)$, then we can always take $M=1$.
2. Note that this theorem gives some idea why Maclaurin approximations get better by using more terms. As n gets bigger, the fraction $\frac{M}{(n+1)!}$ will often get smaller. Why? Because $(n+1)!$ gets really big. Is it true then that $\frac{M}{(n+1)!}$ always get smaller for very large n? Well, to be rigorous, M might also increase with n. But those cases are rare for functions we deal with in general.
3. Note, we are often interested in bounding $|R_n^a f(x)|$ on an interval. In such a case we replace $|x-a|^{n+1}$ by its absolute max on the interval. In other words, if the interval is $I=[c,d]$, we shall replace $|x-a|^{n+1}$ by $|c-a|^{n+1}$ or $|d-a|^{n+1}$, whichever is bigger.

We consider here two simple cases of application of the error estimates.

6.3.1 How to compute e in a few decimal places?

Note that $e=f(1)$ where $f(x)=e^x$. We computed $T_n f(x)$ before. If you set $x=1$, then you get $e=f(1)=T_n f(1)+R_n f(1)$, and thus, taking $n=8$,

$$e = 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\frac{1}{6!}+\frac{1}{7!}+\frac{1}{8!}+R_8(1).$$

By Lagrange's formula there is a ξ between 0 and 1 such that

$$R_8(1)=\frac{f^{(9)}(\xi)}{9!}\,1^9=\frac{e^\xi}{9!}.$$

We do not really know where ξ is, but, since it lies between 0 and 1, we know that $1 < e^{\xi} < e$. So, the remainder term $R_8(1)$ is positive and no more than $e/9!$. Estimating $e < 3$, we find

$$\frac{1}{9!} < R_8(1) < \frac{3}{9!} \approx (120000)^{-1}.$$

Note that we have been able to estimate a **lower bound** and **upper bound** for $R_8 f(1)$. Thus we see that

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{7!} + \frac{1}{8!} + \frac{1}{9!} < e < 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{7!} + \frac{1}{8!} + \frac{3}{9!}$$

or, in decimals,

$$2.718\,281\ldots < e < 2.718\,287\ldots$$

6.3.2 How good is the approximation of $T_4 f$ for $f(x) = \cos(x)$?

For $f(x) = \cos(x)$, let's look at the approximation $T_4 f(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!}$ on the interval $[-\frac{1}{2}, \frac{1}{2}]$. We might ask within what tolerance does $T_4 f$ approximate $\cos x$ on that interval? To answer this, recall that the error term for $T_4 f$ is

$$R_4 f(x) = \frac{-\sin\xi}{5!}\, x^5.$$

For x in the indicated interval, we want to know the **worst-case scenario** for the size of this error. A sloppy but good and simple **estimate** on $\sin\xi$ is that $|\sin\xi| \leqslant 1$, regardless of what ξ is. This is a very happy kind of estimate because it is not so bad and because it does not depend at all upon x. The biggest that x^5 can be is $(\frac{1}{2})^5 \approx 0.03$. Then the **error is estimated as**

$$|R_4 f(x)| = |\frac{-\sin\xi}{5!} x^5| \leqslant \frac{1}{2^5 \cdot 5!} \leqslant 0.0003.$$

This is not so bad at all! We could have been a little clever here, taking advantage of the fact that $T_4 f = T_5 f$. Thus, we are actually entitled to use the remainder term $R_5 f$. This typically will give a better outcome, namely

$$R_5 f(x) = -\frac{\cos\xi}{6!}\, x^6.$$

Again, in the **worst-case scenario** $|-\cos\xi| \leqslant 1$. And still $|x| \leqslant \frac{1}{2}$, so we have the **error estimate**

$$|R_5 f(x)| = |\frac{\cos\xi}{6!} x^6| \leqslant \frac{1}{2^6 \cdot 6!} \leqslant 0.000022.$$

This is less than a tenth as much as in the first version.

But what happened here? Are there two different answers to the question of how well that polynomial approximates the cosine function on that interval? Of course not. Rather, there were two approaches taken by us to estimate how well it approximates cosine. In fact, we still do not know the **exact** error! The point is that the second estimate (being a little wiser) is **closer** to the truth than the first. The first estimate is **true**, but is a **weaker** assertion than we are able to make if we try a little harder.

This already illustrates the point that 'in real life' there is often no single 'right' or 'best' estimate of an error, in the sense that the estimates that we can obtain by practical procedures may not be perfect, but represent a trade-off between time, effort, cost, and other priorities.

6.3.3 Error in the Approximation $\sin x \approx x$

In many calculations involving $\sin x$ for small values of x one makes the simplifying approximation $\sin x \approx x$, justified by the known limit

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

How big is the error in this approximation?

To answer this question, we use Lagrange's formula for the remainder term. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \sin x$. Its Maclaurin polynomial of order 1 is

$$T_1 f(x) = x,$$

of remainder

$$R_1 f(x) = \frac{f''(\xi)}{2!} x^2 = -\tfrac{1}{2} \sin \xi \cdot x^2$$

for some ξ between 0 and x. As always with Lagrange's remainder term, we don't know where ξ is precisely, so we estimate the remainder term. As we did several time before, $|\sin \xi| \leq 1$, $\forall \xi \in \mathbb{R}$. Hence, the remainder term is bounded by

$$|R_1 f(x)| \leq \tfrac{1}{2} x^2, \tag{6.6}$$

and so we find that

$$x - \tfrac{1}{2} x^2 \leq \sin x \leq x + \tfrac{1}{2} x^2. \tag{6.7}$$

Example. Now we could fix the maximum error we are seeking and ask **how small must we choose x to be sure that the approximation $\sin x \approx x$ is not off by more than, say, 1%?** If we want the error to be less than 1% of the estimate (it is a **relative error**, then we should require $\frac{1}{2} x^2$ to be less than 1% of $|x|$, that is $|x| < 0.02$.

Recall that because $T_1 f = T_2 f$, the error term $R_1 f$ can be estimated using $R_2 f$, that is

$$|R_2 f(x)| \leq \tfrac{1}{6} |x|^3$$

instead of $\frac{1}{2} x^2$ as in (6.6). This means that (6.7) becomes

$$x - \tfrac{1}{6} |x|^3 \leq \sin x \leq x + \tfrac{1}{6} |x|^3. \tag{6.8}$$

Now, is that better? Think about it. It is actually better for **'small'** x, but **worse** for 'large' x.

6.4 Estimating Integrals

Thinking simultaneously about the difficulty (or impossibility) of 'direct' symbolic integration of complicated expressions, by contrast to the ease of integration of **polynomials**, we might hope to get some mileage out of **integrating and differentiating** Taylor polynomials.

Example. We know that

$$\int_0^T \frac{dx}{1-x} = [-\ln(1-x)]_0^T = -\ln(1-T). \tag{6.9}$$

On the other hand, the Maclaurin polynomial of $f(x) = \dfrac{1}{1-x}$ is

$$T_n f(x) = 1 + x + x^2 + \ldots + x^n. \tag{6.10}$$

Replacing into (6.9), we could obtain

$$\int_0^T (1 + x + x^2 + \ldots + x^n)\, dx = \left[x + \frac{x^2}{2} + \ldots + \frac{x^{n+1}}{(n+1)}\right]_0^T = T + \frac{T^2}{2} + \ldots + \frac{T^{n+1}}{(n+1)}.$$

For $x < 1$, let $F(x) = -\ln(1-x)$. Putting the previous results together (and changing the variable back to 'x'), we get the 'formula':

$$T_n F(x) = x + \frac{x^2}{2} + \ldots + \frac{x^{n+1}}{(n+1)}.$$

For the moment we do not worry about what happens to the error term for the Taylor polynomial. This little computation has several useful interpretations.

1. First, we obtained a Taylor polynomial for $-\ln(1-T)$ from that of the geometric series in (6.10), without going to the trouble of recomputing derivatives.
2. Second, from a different perspective, we have an expression for the integral

   $$\int_0^T \frac{dx}{1-x}$$

   without necessarily mentioning the logarithm: that is,

   $$\int_0^T \frac{dx}{1-x} \approx T + \frac{T^2}{2} + \cdots + \frac{T^{n+1}}{(n+1)}$$

   for arbitrary large n.

Being a little more careful, we would like to keep track of the error term in the example so we could be more sure of what we claim. We have

$$\frac{1}{1-x} = 1 + x + x^2 + \ldots + x^n + \frac{1}{(n+1)} \frac{1}{(1-\xi)^{n+1}} x^{n+1}$$

for some ξ between 0 and x (and also depending upon x and n). One way to avoid having the term $\frac{1}{(1-\xi)^{n+1}}$ to 'blow up' on us, is to keep x itself in the range $[0,1)$ so that ξ is in the range $[0,x) \subseteq [0,1)$, keeping ξ away from 1. To do this we might demand that $0 \leqslant T < 1$. For simplicity, and to illustrate the point, we take $0 \leqslant T \leqslant \frac{1}{2}$. Then, in the **worst-case scenario**,

$$|\frac{1}{(1-\xi)^{n+1}}| \leqslant \frac{1}{(1-\frac{1}{2})^{n+1}} = 2^{n+1}.$$

Thus, **integrating the error term,** we have

$$\begin{aligned} |\int_0^T \frac{1}{n+1}\frac{1}{(1-\xi)^{n+1}}x^{n+1}\,dx| &\leqslant \int_0^T \frac{2^{n+1}}{n+1}x^{n+1}\,dx = \frac{2^{n+1}}{n+1}\int_0^T x^{n+1}\,dx \\ &= \frac{2^{n+1}}{n+1}\left[\frac{x^{n+2}}{n+2}\right]_0^T = \frac{2^{n+1}T^{n+2}}{(n+1)(n+2)}. \end{aligned}$$

Since we have required $0 \leqslant T \leqslant \frac{1}{2}$, we actually have

$$|\int_0^T \frac{1}{n+1}\frac{1}{(1-c)^{n+1}}x^{n+1}\,dx| \leqslant \frac{2^{n+1}T^{n+2}}{(n+1)(n+2)} \leqslant \frac{2^{n+1}(\frac{1}{2})^{n+2}}{(n+1)(n+2)} = \frac{1}{2(n+1)(n+2)}.$$

That is, we have

$$|-\log(1-T) - [T + \frac{T^2}{2} + \ldots + \frac{T^n}{n}]| \leqslant \frac{1}{2(n+1)(n+2)}$$

for all T in the interval $[0, \frac{1}{2}]$. Actually, we had obtained

$$|-\log(1-T) - [T + \frac{T^2}{2} + \ldots + \frac{T^n}{n}]| \leqslant \frac{2^{n+1}T^{n+2}}{(n+1)(n+2)}$$

and the latter expression shrinks rapidly as T approaches 0.

Next we return to another example from the previous lecture.

Example. Let $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \cos x$. We use $T_3^a f$ about $a = \pi/2$ to estimate

$$\int_{\pi/3}^{\pi} \cos x\,dx,$$

and compare the estimation with the actual answer. The polynomial is

$$T_3^{\pi/2} f(x) = -(x-\pi/2) + \frac{(x-\pi/2)^3}{3!}.$$

We approximate the integral by

$$\int_{\pi/3}^{\pi} \cos x\,dx \approx \int_{\pi/3}^{\pi} \Big(-(x-\pi/2) + \frac{(x-\pi/2)^3}{3!}\Big)\,dx = \Big[-\frac{(x-\pi/2)^2}{2} + \frac{(x-\pi/2)^4}{24}\Big]_{\pi/3}^{\pi} \approx -0.846085.$$

The exact answer is

$$\int_{\pi/3}^{\pi} \cos x\,dx = \Big[\sin x\Big]_{\pi/3}^{\pi} = -\frac{\sqrt{3}}{2} \approx -0.866025.$$

The approximate answer is quite close to the exact answer. But could we guarantee this? The error could be evaluated from the integral of the remainder term $R_3^{\pi/2} f = \dfrac{f^{(4)}(\xi)}{4!}(x-\pi/2)^4$. Because each derivative of cos is bounded by 1 (determine why?), we get

$$\int_{\pi/3}^{\pi} |R_3^{\pi/2} f(x)|\, dx \leq \int_{\pi/3}^{\pi} \frac{|(x-\pi/2)|^4}{4!}\, dx \leq \frac{[(x-\pi/2)^5]_{\pi/3}^{\pi}}{5!} \leq \frac{244\pi^5}{933120} \approx 0.08.$$

We see that our predicted error is about four times larger than the real value. This is not a contraction, but we could have been smarter. Note that because cos an **odd** function **about** $\boldsymbol{x = \pi/2}$, $T_4^{\pi/2} f(x) = T_3^{\pi/2} f(x)$, and so we could use the remainder $R_4^{\pi/2} f(x)$. In that case, we get the estimate

$$\int_{\pi/3}^{\pi} |R_4^{\pi/2} f(x)|\, dx \leq \int_{\pi/3}^{\pi} \frac{|(x-\pi/2)|^5}{5!}\, dx \leq \frac{[(x-\pi/2)^6]_{\pi/3}^{\pi}}{6!} \leq \frac{91\pi^6}{6^6 \cdot 90} \approx 0.020835,$$

which is basically the error we actually got.

Of course, we would always compute this integral by working out the exact answer, since it is much simpler than doing the approximation in this case. But, more generally, the first few terms of a Taylor expansion can be used as an approximation and might help us approximately determine integrals of functions which cannot be integrated exactly, for instance, in Question 9 in Exercise Sheet 5a,

$$\int_0^{0.5} \sqrt{1+x^4}\, dx.$$

There are no simple exponential or trigonometric functions giving the integral of that square root. And so, to evaluate it, we need some approximation, either numerical (like the Trapezium Rule) or using a Taylor polynomial (or series).

We now give an alternative Taylor estimate using an **integral formulation** for the error.

Theorem 6.6 (Taylor's Theorem Revisited)**.** *Let $f : [a,b] \to \mathbb{R}$ be $n+1$ times differentiable on $[a,b]$. Let $R_n^a f(x) = f(x) - T_n^a f(x)$ for $x \in [a,b]$, then,*

$$|R_n^a f(x)| = \left| \int_a^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n\, dt \right| \leq \frac{M|x-a|^{n+1}}{(n+1)!}, \qquad (6.11)$$

where M is the maximum value of $|f^{(n+1)}(t)|$ on $[a,x]$.

Proof. For $t \in [a,x]$, let

$$S(t) = R_n^t f(x) = f(x) - T_n^t f(x) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(t)}{k!}(x-t)^k.$$

We claim that

$$S'(t) = -\frac{f^{(n+1)}(t)}{n!}(x-t)^n. \qquad (6.12)$$

We have

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{f^{(k)}(t)}{k!}(x-t)^k = \frac{f^{(k+1)}(t)}{k!}(x-t)^k - \frac{f^{(k)}(t)}{k!}k(x-t)^{k-1}.$$

So,

$$\begin{aligned} S'(t) &= -\sum_{k=0}^{n} \frac{f^{(k+1)}(t)}{k!}(x-t)^k + \sum_{k=0}^{n} \frac{f^{(k)}(t)}{k!}k(x-t)^{k-1} \\ &= -\sum_{k=0}^{n} \frac{f^{(k+1)}(t)}{k!}(x-t)^k + \sum_{k=1}^{n} \frac{f^{(k)}(t)}{(k-1)!}(x-t)^{k-1}. \end{aligned}$$

Now we work with the second sum and manipulate it so that the index begins at 0. Set $j = k - 1$. As k runs from 1 to n, j runs from 0 to $n-1$. Substituting $k = 1 + j$ gives

$$S'(t) = -\sum_{k=0}^{n} \frac{f^{(k+1)}(t)}{k!}(x-t)^k + \sum_{j=0}^{n-1} \frac{f^{(j+1)}(t)}{j!}(x-t)^j.$$

But now these two sums are the same except that the first one has an extra term. Consequently most of the terms cancel and we are left with (6.12).

Because

$$S(t) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(t)}{k!}(x-t)^k = f(x) - f^{(0)}(t) - \sum_{k=1}^{n} \frac{f^{(k)}(t)}{k!}(x-t)^k.$$

Hence,

$$S(x) = f(x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!}(x-x)^k = 0,$$

taking care of the initial condition. Now, for any function S,

$$\int_a^x S'(t)\,dt = \big[S(t)\big]_a^x = S(x) - S(a).$$

But $S(a) = R_n^a(x)$. Because $S(x) = 0$,

$$R_n^a f(x) = \int_a^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n\,dt.$$

So,

$$|R_n^a f(x)| = \Big| \int_a^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n\,dt \Big|.$$

Now, since $t \in [a, x]$, $|f^{(n+1)}(t)| \leq M$. So

$$\begin{aligned} |R_n^a f(x)| = \Big| \int_a^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n\,dt \Big| &\leq \Big| \int_a^x \frac{M(x-t)^n}{n!}\,dt \Big| \\ &= M \Big| \Big[-\frac{(x-t)^{n+1}}{(n+1)!} \Big]_a^x \Big| = \frac{M|x-a|^{n+1}}{(n+1)!}. \end{aligned}$$

□

6.5 Extra curricular material: Estimating n for Taylor Polynomials

In this section we look at the issue of determining n in our problems.

Example. Find n such that the Maclaurin polynomial $T_n f(0.5)$ for $f(x) = \sin(x)$ would approximate $\sin(0.5)$ with an error less than 10^{-5}.

using the now familiar estimates for f, the error terms are bounded by

$$|R_n f(x)| = \frac{|x|^{n+1}}{(n+1)!}.$$

Therefore we are looking to find n such that

$$\frac{1}{(n+1)!}(1/2)^{n+1} \leqslant 10^{-5}.$$

Now, this is an **inequality for n**, that cannot be solved exactly. So, we have to guess and check, more or less cleverly. We check only even values for n since there are no even terms in $\sin(x)$!

$$n = 4: \qquad \tfrac{1}{5!}(.5)^5 = 2.6 \times 10^{-4},$$

$$n = 6: \qquad \tfrac{1}{7!}(.5)^7 = 1.55 \times 10^{-6}.$$

Thus $T_5 f(0.5)$ gives an error which is small enough.

Now we look at the most difficult question about accuracy:

Given a function, given a fixed point, given an interval around that fixed point, and given a required tolerance, find **how many terms** must be used in the Taylor expansion to approximate the function to within the required tolerance on the given interval.

Example. For example, let's get a Taylor polynomial approximation to $f(x) = e^x$ which is within 0.001 on the interval $[-\frac{1}{2}, +\frac{1}{2}]$. Recall that

$$T_n f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \frac{e^c}{(n+1)!}x^{n+1}$$

for some ξ between 0 and x, and where we do not yet know what we want n to be. It is very convenient here that the n-th derivative of e^x is still just e^x! We are wanting to choose n large-enough to guarantee that

$$|\frac{e^\xi}{(n+1)!}x^{n+1}| \leqslant 0.001$$

for all x in that interval.

The error term is estimated as follows, by thinking about the **worst-case scenario** for the sizes of the parts of that term: we know that the exponential function is increasing along the whole real line, so in any event ξ lies in $[-\frac{1}{2}, +\frac{1}{2}]$ and

$$|e^\xi| \leqslant e^{1/2} \leqslant 2$$

(where we've not been too fussy about being accurate about how big the square root of e is!). And for x in that interval we know that

$$|x^{n+1}| \leqslant (\frac{1}{2})^{n+1}$$

So we are wanting to **choose n large-enough** to guarantee that

$$|\frac{e^{\xi}}{(n+1)!}(\frac{1}{2})^{n+1}| \leqslant 0.001.$$

Since

$$|\frac{e^{\xi}}{(n+1)!}(\frac{1}{2})^{n+1}| \leqslant \frac{2}{(n+1)!}(\frac{1}{2})^{n+1}.$$

we can be confident of the desired inequality if we can be sure that

$$\frac{2}{(n+1)!}(\frac{1}{2})^{n+1} \leqslant 0.001.$$

That is, we want to 'solve' for n in the inequality

$$\frac{2}{(n+1)!}(\frac{1}{2})^{n+1} \leqslant 0.001.$$

There is no genuine formulaic way to 'solve' for n to accomplish this. Rather, we just evaluate the left-hand side of the desired inequality for larger and larger values of n until (hopefully!) we get something smaller than 0.001. So, trying $n = 3$, the expression is

$$\frac{2}{(3+1)!}(\frac{1}{2})^{3+1} = \frac{1}{12 \cdot 16}$$

which is more like 0.01 than 0.001. So just try $n = 4$:

$$\frac{2}{(4+1)!}(\frac{1}{2})^{4+1} = \frac{1}{60 \cdot 32} \leqslant 0.00052$$

which is better than we need.

The conclusion is that we needed to take the Taylor polynomial of degree $n = 4$ to achieve the desired tolerance along the whole interval indicated. Thus, the polynomial

$$1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4}$$

approximates e^x to within 0.00052 for x in the interval $[-\frac{1}{2}, \frac{1}{2}]$.

Yes, such questions can easily become very difficult. And, as a reminder, there is no real or genuine claim that this kind of approach to polynomial approximation is 'the best'. It will depend on what we want to do.

Summary of Chapter 5

We have seen:

- the definition of a **series**;
- the **geometric** and **Harmonic** series;
- how to sum **telescoping sums**;
- an example of a series where the **terms tend to zero but the series diverges**;
- **three tests** for determining whether a series is convergent;
- that we must **check carefully** that it is appropriate to use either of the tests.

6.6 Exercise Sheet 5

6.6.1 Exercise Sheet 5a

1. Let $f(x) = \cos(x)$, how well does $T_6 f$ approximate $\cos x$ on the intervals
   (a) $[-0.1, 0.1]$?
   (b) $[-1, 1]$?
   (c) $[-\frac{\pi}{2}, \frac{\pi}{2}]$?

2. (*Computing the cube root of* 9) The cube root of 8 is 2, and 9 is 'only' one more than 8. So you could try to compute $\sqrt[3]{9}$ by viewing it as $\sqrt[3]{8+1}$.
   (a) Let $f(x) = \sqrt[3]{8+x}$. Find $T_2 f$, and estimate the error $|\sqrt[3]{9} - T_2 f(1)|$.
   (b) Repeat part (i) for $T_3 f$ and compare.

3. Follow the method of Question 2 to compute $\sqrt{10}$ within an error of 10^{-3} using a polynomial approximation.

4. For what range of $x \geqslant 25$ does $5 + \dfrac{1}{10}(x - 25)$ approximate $\sqrt{x}$ to within 0.001?

5. (a) Let $f(t) = \sin t$. Compute $T_2 f$ and give an upper bound for $R_2 f$ for $t \in [0, 0.5]$.
   (b) Use part (i) to approximate $\int_0^{0.5} \sin(x^2)\, dx$, and give an upper bound for the error in your approximation.

6. Let $f(t) = e^t$.
   (a) Find $T_2^a f$ for $a = 0$, $a = 1/2$ and $a = 1$.
   (b) Use each of the previous Taylor polynomials approximation of f to give estimates for the integral
   $$\int_0^1 e^{x^2} dx, \tag{6.13}$$
   and compare them.
   (c) Suppose instead we used $T_5^a f$ for the same values of a. Estimate the integral (6.13) and give an upper bound for the errors.

7. (a) Approximate $\int_0^{0.1} \arctan x\, dx$ and estimate the error in your approximation by analyzing $T_2 f(t)$ and $R_2 f(t)$ where $f(t) = \arctan t$.
   (b) Estimate $\int_0^{0.1} \arctan x\, dx$ with an error of less than 0.001.

8. Approximate $\int_0^{0.1} x^2 e^{-x^2} dx$ and estimate the error in your approximation by analyzing $T_3 f(t)$ and $R_3 f(t)$ where $f(t) = te^{-t}$.

9. Estimate $\int_0^{0.5} \sqrt{1 + x^4}\, dx$ with an error of less than 10^{-4}.

6.6.2 Exercise Sheet 5b

1. For what range of values of x is $x - \frac{x^3}{6}$ within 0.01 of $\sin x$?

2. Only consider $-1 \le x \le 1$. For what range of values of x **inside this interval** is the polynomial $1 + x + x^2/2$ within .01 of e^x?

3. On how large an interval around 0 is $1 - x$ within 0.01 of $1/(1 + x)$?

4. On how large an interval around 100 is $10 + \frac{x - 100}{20}$ within 0.01 of $\sqrt{x}$?

5. Determine how many terms are needed in order to have the corresponding Taylor polynomial approximate e^x to within 0.001 on the interval $[-1, +1]$.

6. Determine how many terms are needed in order to have the corresponding Taylor polynomial approximate $f(x) = \cos x$ to within the tolerance ϵ on the interval $[a, b]$ for the following cases:

   (a) $\epsilon = 0.001$, $[a, b] = [-1, +1]$;

   (b) $\epsilon = 0.001$, $[a, b] = [-\frac{\pi}{2}, \frac{\pi}{2}]$;

   (c) $\epsilon = 0.001$, $[a, b] = [-0.1, +0.1]$.

7. Approximate $\sqrt{e}$ to within 0.01 by using a Maclaurin polynomial with remainder term.

8. Approximate $\sqrt{101}$ to within 10^{-15} using a Taylor polynomial with remainder term.

   Note: do NOT add up the finite sum you get! One point here is that most hand calculators do not easily give 15 decimal places.

9. Find the 4-th degree Maclaurin polynomial for the function $f(x) = \sin x$. Estimate the error for $|x| < 1$.

10. Let $f(x) = \cos x$.

    (a) Find the fourth degree Maclaurin polynomial for f and estimate the error for $|x| < 1$.

    (b) Find the eighth degree Maclaurin polynomial for f and estimate the error for $|x| < 1$.

    (c) Now, find the ninth degree Maclaurin polynomial for f. What is the error for $|x| \le 1$?

6.6.3 Miscellaneous Exercises

Question

1. If

$$T_n f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

is the Taylor polynomial of a function $y = f(x)$, then what is the Taylor polynomial of its derivative $f'(x)$?

Theorem 6.7. *The Taylor polynomial of degree $n-1$ of $f'(x)$ is given by*

$$T_{n-1}\{f'(x)\} = a_1 + 2a_2 x + \cdots + n a_n x^{n-1}.$$

In other words, "the Taylor polynomial of the derivative is the derivative of the Taylor polynomial."

Proof. Let $g(x) = f'(x)$. Then $g^{(k)}(0) = f^{(k+1)}(0)$, so that

$$\begin{aligned} T_{n-1} g(x) &= g(0) + g'(0)x + g^{(2)}(0)\frac{x^2}{2!} + \cdots + g^{(n-1)}(0)\frac{x^{n-1}}{(n-1)!} \\ &= f'(0) + f^{(2)}(0)x + f^{(3)}(0)\frac{x^2}{2!} + \cdots + f^{(n)}(0)\frac{x^{n-1}}{(n-1)!} \qquad (\$) \end{aligned}$$

On the other hand, if $T_n f(x) = a_0 + a_1 x + \cdots + a_n x^n$, then $a_k = f^{(k)}(0)/k!$, so that

$$k a_k = \frac{k}{k!} f^{(k)}(0) = \frac{f^{(k)}(0)}{(k-1)!}.$$

In other words,

$$1 \cdot a_1 = f'(0),\ 2a_2 = f^{(2)}(0),\ 3a_3 = \frac{f^{(3)}(0)}{2!},\ \text{etc.}$$

So, continuing from ($) you find that

$$T_{n-1}\{f'(x)\} = T_{n-1} g(x) = a_1 + 2a_2 x + \cdots + n a_n x^{n-1}$$

as claimed. □

2. For the following functions, find an approximation to f by a Taylor polynomial of degree n about a number a.

Use Taylor's Theorem to estimate the accuracy of the approximation and compare with the exact value.

(a) $f(x) = x \sin x,\ a = 0,\ n = 4,\ x = 1;$ The Taylor polynomial is $x^2 - \frac{x^4}{6}$. When $x = 1$, the approximation to $x \sin x$ given by the Taylor polynomial is equal to $\frac{5}{6} \approx 0.833$. However the actual value of $1 \times \sin 1$ is approximately 0.841 giving an error of approximately 0.008.

A bound on the error is $\frac{6}{120} = 0.05$. We see that the actual error is much smaller than this.

(full) To compute the Taylor polynomial of degree 4 and a bound on the remainder, we will need the first 5 derivatives of $x \sin x$. Let $f(x) = x \sin x$. Then

$$\begin{aligned} &f^{(1)}(x) = \sin x + x\cos x, \quad &&f^{(2)}(x) = 2\cos x - x\sin x, \quad f^{(3)}(x) = -3\sin x - x\cos x,\\ &f^{(4)}(x) = -4\cos x + x\sin x, \quad &&f^{(5)}(x) = 5\sin x + x\cos x. \end{aligned}$$

So the Taylor polynomial is

$$\sum_{n=0}^{4} \frac{f^{(n)}(0)}{n!} x^n = x^2 - \frac{x^4}{6}.$$

When $x = 1$, the approximation to $x \sin x$ given by the Taylor polynomial is equal to $\frac{5}{6} \approx 0.833$. However the actual value of $1 \times \sin 1$ is approximately 0.841 giving an error of approximately 0.008.

The bound on the error given by Taylor's Theorem is $\frac{M}{5!}1^5$ where M is the absolute maximum of $|f^{(5)}(t)|$ for $t \in [0, 1]$. This value is not easy to find exactly, because it is not easy to maximise $5 \sin x + x \cos x$. However since $|\sin x| \leq 1$ and $|\cos x| \leq 1$ for all x, we know that M is at most 6. So a bound on the error is $\frac{6}{120} = 0.05$. We see that the actual error is much smaller than this.

(b) $f(x) = e^{x^2},\ a = 0,\ n = 3,\ x = 0.5$.

The Taylor polynomial is $1 + x^2$. When $x = 0.5$, the approximation to e^{x^2} given by the Taylor polynomial is equal to $\frac{5}{4} = 1.25$. However the actual value of $e^{0.5^2}$ is approximately 1.284 giving an error of approximately 0.034.

A bound on the error is $\frac{32.101}{24}(0.5)^4 \approx 0.084$. Of course, the actual error is no larger than this.

(full) To compute the Taylor polynomial of degree 3 and a bound on the remainder, we will need the first 4 derivatives of e^{x^2}. Let $f(x) = e^{x^2}$. Then

$$\begin{aligned} &f^{(1)}(x) = 2xe^{x^2}, \quad &&f^{(2)}(x) = (2 + 4x^2)e^{x^2},\\ &f^{(3)}(x) = (12x + 8x^3)e^{x^2}, \quad &&f^{(4)}(x) = (12 + 48x^2 + 16x^4)e^{x^2}. \end{aligned}$$

So the Taylor polynomial is

$$\sum_{n=0}^{3} \frac{f^{(n)}(0)}{n!} x^n = 1 + x^2.$$

When $x = 0.5$, the approximation to e^{x^2} given by the Taylor polynomial is equal to $\frac{5}{4} = 1.25$. However the actual value of $e^{0.5^2}$ is approximately 1.284 giving an error of approximately 0.034.

The bound on the error given by Taylor's Theorem is $\frac{M}{4!}(0.5)^4$ where M is the absolute maximum of $|f^{(4)}(t)|$ for $t \in [0, 0.5]$. Since $f^{(4)}$ is increasing on $[0, \infty)$, we have $M = f^{(4)}(0.5) \approx 32.101$. So a bound on the error is $\frac{32.101}{24}(0.5)^4 \approx 0.084$. Of course, the actual error is no larger than this.

Exercise 6.8. *For the following functions, find an approximation to f by a Taylor polynomial of degree n about a number a. Use Taylor's Theorem to estimate the accuracy of the approximation and compare with the exact value.*

(a) $f(x) = \sin x,\ a = \pi/6,\ n = 4,\ x = \pi/3$; *(b)* $f(x) = \ln(1+x),\ a = 1,\ n = 3,\ x = 2$.

Exercise 6.9. *Determine the nth degree Taylor polynomial for e^x about $x = 0$ and write down an expression for the remainder.*

Use this to determine n so that the nth degree Taylor polynomial for e^x evaluated at $x = \frac{1}{2}$ gives an approximation to $\sqrt{e}$ for which the error is at most 0.0001.

Compute the approximation and check with the correct answer.

Short Feedback for Exercise Sheet 5a

1. (a) The error is smaller than $2.5 \cdot 10^{-13}$.
   (b) The error is smaller than $2.5 \cdot 10^{-5}$.
   (c) The error is smaller than 10^{-3}.

2. (a) $T_2 f(x) = 2 + \frac{1}{12}x - \frac{1}{288}x^2$ and the error is smaller than $2.5 \cdot 10^{-4}$.
   (b) $T_3 f(x) = 2 + \frac{1}{12}x - \frac{1}{288}x^2 + \frac{5}{10368}x^3$ and the error is smaller than $3.52 \cdot 10^{-5}$. Compare with $T_2 f$ with improve by one digit.

3. Take $f(x) = \sqrt{9+x}$. Then $T_2 f(1) = \frac{683}{216}$ approximate $\sqrt{10}$ within an error of $3 \cdot 10^{-4} < 10^{-3}$.

4. $25 \leq x \leq 26$.

5. (a) $T_2 f(t) = t$ and $|R_2 f(t)| \leq \frac{t^3}{6}$ for $t \in [0, 0.5]$.
   (b) The approximation of the integral is $\frac{1}{24}$ with an error of at most $1/5376 < 2 \cdot 10^{-4}$.

6. (a) $T_2 f(t) = 1 + t + \frac{t^2}{2}$, $T_2^{1/2} f(t) = \sqrt{e}\left(1 + (t - 0.5) + \frac{(t-0.5)^2}{2}\right)$ and $T_2^1 f(t) = e\left(1 + (t-1) + \frac{(t-1)^2}{2}\right)$.
   (b) Integrating over the interval $[0, 1]$ the polynomial approximations, we find 1.4333, 1.4701 and the worse one 1.631.
   (c) With $T_5 f$, the estimate is 1.4625 with error less than $3 \cdot 10^{-4}$.

7. (a) $T_2 f(t) = t$, the integral is approximated by $5 \cdot 10^{-3}$ and the error is less than $3.5 \cdot 10^{-4}$.
   (b) Nothing to do, the previous calculation is OK.

8. The value is $3.3134 \cdot 10^{-4}$ with an error of less than 10^{-5}.

9. Use $T_2 f$, to find 0.5031.

6.6.4 Feedback for Exercise Sheet 5a

1. The error on $[-r,r]$ is bounded by $\frac{r^8}{8!}$.

   (a) Here, $r = 1/10$, and so the error is smaller than $\frac{1}{10^8 8!} \leq 2.5\ 10^{-13}$.

   (b) Here, $r = 1$, and so the error is smaller than $\frac{1}{8!} \leq 2.5\ 10^{-5}$.

   (c) Here, $r = \pi/2$, and so the error is smaller than $\frac{\pi^8}{2^8 8!} \leq 10^{-3}$.

2. (a) The polynomial is $T_2 f(x) = 2 + \frac{1}{12}x - \frac{1}{9\cdot 32}x^2$. Thus, $T_2 f(1) \approx 2.07986111$. The error is

   $$|\sqrt[3]{9} - T_2 f(1)| \leq \frac{10}{27}\cdot 8^{-\frac{8}{3}}\cdot\frac{1}{3!} < 2.5\cdot 10^{-4}.$$

   (b) The $\sqrt[3]{9}$ according to a computer is:

   $$\sqrt[3]{9} \approx 2.08008382305.$$

3. (a) Use Taylor's formula with $f(x) = \sqrt{9+x}$, $n = 1$, to calculate $\sqrt{10}$ approximately. The error is less than $1/216$.

   (b) Repeat with $n = 2$. The error is less than 0.0003.

4. One can easily check that the question ask for the larger range $25 \leq x \leq c$ for which $T_1^{25} f$ approximate f within 0.001. Indeed, for $f(x) = \sqrt{x}$, taking $a = 25$, we have

   $$T_1^{25} f(x) = f(a) + f'(a)(x-a) = \sqrt{25} + \frac{1}{2}\frac{1}{\sqrt{25}}(x-25) = 5 + \frac{1}{10}(x-25).$$

   The corresponding remainder term is $\frac{f''(\xi)}{2!}(x-a)^2$ for some ξ between a and x. Explicitly we have

   $$R_1^{25} f(x) = -\frac{1}{2!}\frac{1}{4}\frac{1}{(c)^{3/2}}(x-25)^2 = -\frac{1}{8}\frac{1}{\xi^{3/2}}(x-25)^2,$$

   where $25 \leq \xi \leq x$. Because the three-halves-power function is **increasing**, we have

   $$25^{3/2} \leqslant \xi^{3/2} \leqslant x^{3/2}$$

   Taking inverses (with positive numbers) reverses the inequalities: we have

   $$25^{-3/2} \geqslant \xi^{-3/2} \geqslant x^{-3/2}$$

   So, **in the worst-case scenario**, the value of $\xi^{-3/2}$ is at most $25^{-3/2} = 1/125$. Taking the absolute value for the remainder term (in order to talk about error),

   $$|R_1 f^{25} f(x)| = |\frac{1}{8}\frac{1}{\xi^{3/2}}(x-25)^2| \leq |\frac{1}{8}\frac{1}{125}(x-25)^2|.$$

To give a range of values of x for which we can be sure that the error is never larger than 0.001 **based upon our estimate**, we solve the inequality

$$|\frac{1}{8}\frac{1}{125}(x-25)^2| \leqslant 0.001$$

(with $x \geqslant 25$). Multiplying out by the denominator of $8 \cdot 125$ gives

$$|x-25|^2 \leqslant 1$$

so the solution is $25 \leqslant x \leqslant 26$.

5. (a) The Taylor series is

$$\sin(t) = t - t^3/6 + \cdots$$

and the order one and two Taylor polynomial is the same $p(t) = t$. For any t there is a ζ between 0 and t with

$$\sin(t) - p(t) = \frac{f^{(3)}(\zeta)}{3!}t^3$$

When $f(t) = \sin(t)$, $|f^{(n)}(\zeta)| \leq 1$ for any n and ζ. Consequently,

$$|\sin(t) - p(t)| \leq \frac{t^3}{3!}$$

for nonnegative t.

(b) Hence

$$\begin{aligned}\left|\int_0^{\frac{1}{2}} \sin(x^2)\,dx - \int_0^{\frac{1}{2}} p(x^2)\,dx\right| &\leq \int_0^{\frac{1}{2}} |\sin(x^2) - p(x^2)|\,dx \\ &\leq \int_0^{\frac{1}{2}} \frac{x^6}{3!}\,dx = \frac{(1/2)^7}{3!\,7} = \epsilon\end{aligned}$$

Since $\displaystyle\int_0^{\frac{1}{2}} p(x^2)\,dx = \frac{(1/2)^3}{3} = A$ (the approximate value) we have that

$$A - \epsilon \leq \int_0^{\frac{1}{2}} \sin(x^2)\,dx \leq A + \epsilon$$

6. (a) Because all the derivatives of f are equal to f, $T_2 f(t) = 1 + t + \dfrac{t^2}{2}$, $T_2^{1/2} f(t) = \sqrt{e}\left(1 + (t-0.5) + \dfrac{(t-0.5)^2}{2}\right)$ and $T_2^1 f(t) = e\left(1 + (t-1) + \dfrac{(t-1)^2}{2}\right)$.

(b) The respective polynomial estimates for e^{x^2} are $1+x^2+\dfrac{x^4}{2}$, $\sqrt{e}\left(\dfrac{5}{8} + \dfrac{x^2}{2} + \dfrac{x^4}{2}\right)$ and $\dfrac{e}{2}(1+x^4)$, respectively. Integrating over the interval $[0, 1]$ the polynomial estimates, we find $\dfrac{43}{30} = 1.4333$, 1.4701 and the worse one 1.631.

(c) Note: You need not find $p(t)$ or the integral $\int_0^1 p(x^2)\ dx$. The error for $T_5 f$ is smaller than $\frac{3}{6! \cdot 13}$.

7. (a)

   (b)

8.

9.

Short Feedback for Exercise Sheet 5b

1.

For Question 10a:

The derivatives of f satisfy $f(x) = f^{(4)}(x) = \cos x$, $f^{(1)}(x) = f^{(5)}(x) = -\sin x$, $f^{(2)}(x) = -\cos x$ and $f^{(3)}(x) = \sin x$. And so $f(0) = f^{(4)}(0) = 1$, $f^{(1)}(0) = f^{(3)}(0) = 0$, $f^{(2)}(0) = -1$ and hence the fourth degree Maclaurin polynomial of f is

$$T_4 f(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}.$$

The error is

$$R_4 f(x) = \frac{f^{(5)}(\xi)}{5!} x^5 = \frac{(-\sin \xi)}{5!} x^5$$

for some ξ between 0 and x. As $|\sin \xi| \leqslant 1$, we have

$$|R_4(x)| \leqslant \frac{|x^5|}{5!} < \frac{1}{5!}$$

for $|x| < 1$. Remark that since the fourth and fifth order Maclaurin polynomial for f are the same, $R_4(x) = R_5(x)$. It follows that $\frac{1}{6!}$ is also an upperbound.

Chapter 7

Power and Taylor Series

7.1 Lecture 19: About Taylor and Maclaurin Series

Let $I \subseteq \mathbb{R}$ be an open interval and $f : I \to \mathbb{R}$ be **infinitely differentiable** at $x = a \in I$. We can construct the Taylor polynomials $T_n^a f$ of f of **any** degree n.

Definition 7.1. *Let $I \subset \mathbb{R}$ be an open interval and $f : I \to \mathbb{R}$ be an infinitely differentiable function at $a \in I$. The* **Taylor SERIES $T_\infty^a f$ at a point a of f** *is the series*

$$T_\infty^a f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n. \tag{7.1}$$

When $a = 0$, we denote the Taylor series by **$T_\infty f$** *and call it the* **Maclaurin series at a point a** *of* **f**.

7.1.1 Some Special Examples of Maclaurin Series

We now list expansions you should be aware of. These Maclaurin series can be computed directly from the definition by repeatedly differentiating the rule:

$$\begin{aligned}
T_\infty e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots, \\
T_\infty \sin(x) &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{6} + \frac{x^5}{120} + \dots, \\
T_\infty \cos(x) &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{24} + \dots, \\
T_\infty\left(\frac{1}{1-x}\right) &= \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 \dots, \qquad \text{(Geometric Series)}, \\
T_\infty(\ln(1+x)) &= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} + \dots.
\end{aligned}$$

Another function whose Maclaurin polynomial/series you should know is $f(x) = (1+x)^\alpha$, where $\alpha \in \mathbb{R}$ is a constant. We have already seen it at Level 1. You can compute $T_n f(x)$ directly from the definition, and, when you do this, you find

$$T_n f(x) = 1 + \alpha x + \frac{\alpha(\alpha-1)}{1\cdot 2}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{1\cdot 2\cdot 3}x^3 + \cdots + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{1\cdot 2\cdots n}x^n. \tag{7.2}$$

This formula is called **Newton's binomial formula**. The coefficient of x^n is called a **binomial coefficient**, and it is written

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}. \tag{7.3}$$

When α is an integer, $\binom{\alpha}{n}$ is also called **α choose n**. When α is a positive integer, the binomial coefficients are just the numbers in **Pascal's triangle**. When $\alpha = -1$, the binomial formula is the Geometric Series.

7.1.2 Convergence Issues about Taylor Series

We have created **infinite series $T^a_\infty f$**. And so, we have usual mathematical questions.

1. **What sense can we make of the series $T^a_\infty f$?**
2. **What is the relation between the series $T^a_\infty f$ and the limit of $T^a_n f$ as $n \to \infty$?**
3. When they exist, **what are the properties of the series and the limit?**

To approach those questions, we first remind you of the error estimates for Taylor polynomials.

Reminder about Taylor Polynomials

Recall that, for $x \in [a,b]$, we call $R^a_n(x) = f(x) - T^a_n f(x)$, the **remainder of order n at $x = a$**. We had the **Lagrange formula** for the remainder when f has $n+1$ continuous derivatives:

$$R^a_n f(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$$

for some ξ between a and x (ξ depending on x). We also derived an alternative formula:

$$R^a_n(x) = \int_a^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n\, dt.$$

When we work for many x simultaneously, the following estimate for the remainder will prove useful:

$$\left|R^a_n(x)\right| \le \frac{M|b-a|^{n+1}}{(n+1)!}, \quad \forall x \in [a,b],$$

where $|f^{(n+1)}(x)| \le M$ for $x \in [a,b]$.

Convergence May or May Not Occur

We can now look at an explicit example to see that our initial questions are issues that are not necessarily easy to ignore.

Example. Consider $f : \mathbb{R}\backslash\{1\} \to \mathbb{R}$, $f(x) = 1/(1-x)$. From its definition, the Maclaurin expansion of f is $\sum_{n=0}^{\infty} x^n$ and, from the theory of convergence of series, such series converges for $|x| < 1$. From the formula for sum of geometric progressions we have

$$\begin{aligned} f(x) &= \frac{1}{1-x} = \frac{1 - x^{n+1} + x^{n+1}}{1-x} \\ &= 1 + x + x^2 + \cdots + x^n + \frac{x^{n+1}}{1-x} = T_n f(x) + \frac{x^{n+1}}{1-x}. \end{aligned}$$

The remainder term is

$$R_n f(x) = \frac{x^{n+1}}{1-x},$$

and, when $|x| < 1$, we have

$$\lim_{n\to\infty} |R_n f(x)| = \lim_{n\to\infty} \frac{|x|^{n+1}}{|1-x|} = \frac{\lim_{n\to\infty} |x|^{n+1}}{|1-x|} = \frac{0}{|1-x|} = 0.$$

Thus we have shown that

1. the Maclaurin series of f converges for all $-1 < x < 1$, but not for $|x| \geq 1$, even if the function f is defined for any other point but $x = 1$,
2. the remainder term $R_n f(x)$ valued at any $|x| < 1$ tends to 0 as $n \to \infty$,
3. the Maclaurin series converges to f on $(-1, 1)$.

So, **for $x \in (-1, 1)$**, we can ignore the Taylor series notation and write with confidence

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

7.1.3 Valid Taylor Series Expansions

From the previous example, we see that the following questions are worth asking. Let $I \subset \mathbb{R}$ be an open interval, let $f : I \to \mathbb{R}$ be a function, infinitely differentiable function at $a \in I$.

1. For which $x \in I$ does $T_\infty^a f(x)$ exist (i.e. converge as a series in n, x fixed)?
2. For which $x \in I$ does $T_\infty^a f(x)$ converge to $f(x)$?
3. What are the relations between f, f' and $\int f$ and $T_\infty f$, $T_\infty f'$ and $\int T_\infty f$?

Remarks 7.2. *1. Examples later on will show that we can get converging Taylor series expansion whose sum is not equal to the original function. However this is a fairly rare occurrence for our examples and exercises.*

2. At $x = a$, the Taylor series is always equal to $f(a)$.

3. For many functions, we shall see that their Taylor series converges to $f(x)$ for x in some interval $a - R < x < a + R$ around a.

Definition 7.3. *We shall then say that a function f has a* **valid power series expansion about a** *if $\lim_{n\to\infty} R_n^a(x) = 0$ for x in a neighbourhood of a, that is, when $T_\infty^a(x) = f(x)$.*

There is a simple example showing that everything we can hope for can indeed occur.

Proposition 7.4. *The exponential function $f(x) = e^x$ has a valid Maclaurin expansion, converging for all $x \in \mathbb{R}$.*

Proof. First, remember that $T_\infty e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. To prove that this really is a power series expansion of e^x, valid for all x, we need to prove that $\lim_{n\to\infty} R_n(x) = 0$. Our aim will be to use the Sandwich Theorem to show that

$$\lim_{n\to\infty} |R_n(x)| = 0.$$

We know that $0 \leq |R_n(x)|$ so all we need to do is find an upper bound for $|R_n(x)|$.
Taylor's Theorem says that

$$|R_n(x)| \leq \frac{Mx^{n+1}}{(n+1)!},$$

where M is the absolute maximum of $|f^{(n+1)}(t)|$ on the interval $[0, x]$ or $[x, 0]$ (depending on the sign of x). Now, $f^{(n+1)}(t) = e^t \leq e^{|t|}$, so the absolute maximum of $e^{|t|}$ on the interval $[0, x]$ or $[x, 0]$ is $e^{|x|}$. Therefore $|R_n(x)| \leq \frac{e^{|x|}x^{n+1}}{(n+1)!}$.

We claim that $\lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} = 0$. The quotient of successive terms in the sequence $\left\{\frac{x^{n+1}}{(n+1)!}\right\}_{n=0}^{\infty}$ is $\frac{x}{n+1}$, which means that the sequence **in n** is initially increasing, then its tail is monotonically decreasing for $n > x$. The sequence is bounded below, so it is a simple exercise to show that its limit must be 0 (see Exercise 4 of Sheet 6b). Hence,

$$\lim_{n\to\infty} \frac{e^{|x|}x^{n+1}}{(n+1)!} = \lim_{n\to\infty} e^{|x|} \lim_{n\to\infty} x^{n+1}(n+1)! = e^{|x|} \cdot 0 = 0.$$

So, using the Sandwich Theorem, $\lim_{n\to\infty} R_n(x) = 0$. □

Because $f = f' = \int f$ on $\mathbb{R}$ when f is the exponential function, we get the following result.

Corollary 7.5. *Let $f : \mathbb{R} \to \mathbb{R}$, defined as $f(x) = e^x$. Then $T_\infty f$, $T_\infty f'$ and $T_\infty(\int f)$ are all valid on $\mathbb{R}$ and we obtain each of them from the other by differentiating or integrating the relevant series term per term.*

7.2 Lecture 20: Power Series, Radius of Convergence

Taylor or MacLaurin series belong to a special type of series called **power series**. These are incredibly important in mathematics, both theoretically and for doing numerical calculations.

Definition 7.6. *A* **power series** *is a series of the form*

$$\sum_{n=0}^{\infty} c_n(x-a)^n,$$

where $\{c_n\}_{n=0}^{\infty}$ is a sequence of real numbers.

We think of a and $c_0, \ldots, c_n \ldots$ as being constants and x as a variable. So, a power series is a **function rule** $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$.

Remark 7.7. *Often we work with the special case where $a = 0$, i.e. we use power series of the form*

$$\sum_{n=0}^{\infty} c_n x^n.$$

To define a proper function, we need to define the domain of the power series. The **natural domain of a power series** is the set of values of x for which $f(x)$ converges. We can use the Root or the Ratio Tests to check such convergence.

For the geometric series $\sum_{n=0}^{\infty} x^n$, it was easy to find the domain and it had a simple answer: $(-1, 1)$. The following theorem shows that the domain of a power series, or equivalently the values of x for which it converges, always has this form. Before, let start with a definition and a lemma to indicate that the definition makes sense.

Definition 7.8. *Let $\sum_{n=0}^{\infty} c_n(x-a)^n$ be a power series and let*

$$\rho_1 = \lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right|, \quad \text{and} \quad \rho_2 = \lim_{n\to\infty} |c_n|^{1/n}. \tag{7.4}$$

The **radius of convergence R** *of the series is equal to*

1. *$R = 1/\rho_2$ when $0 < \rho_2 < \infty$,*
2. *$R = 0$ when $\rho_2 = \infty$,*
3. *$R = \infty$ when $\rho_2 = 0$.*

The radius of convergence is defined from ρ_2 that comes from the Root Test because it applies to slightly more cases (see Exercise 10 of Sheet 6a), but they are in general equal.

Lemma 7.9. *In (7.4), when ρ_1 and ρ_2 exist, they are equal.*

Proof. This is quite tricky — it is not examinable. □

Then, the domain of f is determined by its radius of convergence.

Theorem 7.10 (Pointwise convergence of a power series)**.** *Given a power series* $\sum_{n=0}^{\infty} c_n(x-a)^n$*, either:*

1. *the series converges for all* x *when* $R = \infty$*;*
2. *the series converges just for* $x = a$ *when* $R = 0$*;*
3. *the series converges for all* x *with* $|x - a| < R$ *and we need to check what happens for* $x = a + R$ *and* $x = a - R$*.*

Proof. Apply the Root Test to the power series. We need that $\rho_2 \cdot |x| < 1$ for convergence or $\rho_2 \cdot |x| > 1$ for divergence. The proof is completed in Exercise 8 of Sheet 6a. □

Examples.

1. To find the radius of convergence of the power series $\sum_{n=1}^{\infty} \frac{(x-1)^n}{n2^n}$, we need to look at $c_n = \frac{1}{n2^n}$. We evaluate
$$\rho_2 = \lim_{n\to\infty} |c_n|^{1/n} = \frac{1}{2}\lim_{n\to\infty} n^{-1/n} = \frac{1}{2},$$
and so the radius of convergence is $R = 1/\rho_2 = 2$. To test Lemma 7.9,
$$\rho_1 = \lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|} = \lim_{n\to\infty} \frac{n2^n}{(n+1)2^{n+1}} = \lim_{n\to\infty} \frac{(n)}{2(n+1)} = \frac{1}{2} = \rho_2.$$
So, the power series converges when $|x - 1| < 2$ and diverges when $|x - 1| > 2$. To see what happens at the radius of convergence, we replace $x = 3$ and $x = -1$ in the series. In the first case, we get the **divergent harmonic series** $\sum_{n=1}^{\infty} \frac{1}{n}$, in the second case **alternating (convergent) harmonic series** $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$. Therefore the series is well defined on $[-1, 3)$.

2. As we know, the power series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for all $x \in \mathbb{R}$. Indeed, its radius of convergence is $R = \infty$ because
$$\rho_1 = \lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|} = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1} = 0.$$

3. The power series $\sum_{n=0}^{\infty} n!x^n$ converges only for $x = 0$ because
$$\rho_1 = \lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|} = \lim_{n\to\infty} \frac{(n+1)!}{n!} = \lim_{n\to\infty} (n+1) = \infty.$$
The coefficient c_n tends so fast to infinity that no small (but non zero) amount of x can keep the series converging.

7.2.1 Behaviour at the Boundary of the Interval of Convergence

When $0 < R < \infty$ exists, the power series converges when $|x - a| < R$ and diverges when $|x-a| > R$. When $|x-a| = R$, that is, $x = a \pm R$, anything can happen. It needs to be looked at individually. The power series becomes $\sum_{n=1}^{\infty}(-1)^n c_n R^n$. In particular it is quite clear that the theory of **alternating series** is important there. We shall see in the next example (7.5) thereafter that any behaviour can be found at the boundaries of the interval of convergence.

Examples. Let $p \in \mathbb{R}$ be a fixed parameter. Consider the power series

$$\sum_{n=1}^{\infty} \frac{(x-1)^n}{n^p 2^n}. \tag{7.5}$$

What is its radius of convergence (depending on p)? We have:

$$\rho_1 = \lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|} = \lim_{n\to\infty} \frac{n^p 2^n}{(n+1)^p 2^{n+1}} = \frac{1}{2} \lim_{n\to\infty} (1 + 1/n)^p = \frac{1}{2},$$

and so $R = 2$ (independently of p). What happens at the boundary of the interval of convergence? **This will depend on p.** We get the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

at $x = 3$, and

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$$

at $x = -1$. And so, mostly using the Integral Test,

1. when $p > 1$, we have convergence on both sides of the interval, the power series converges on $[-1, 3]$ and diverges outside;

2. when $0 < p \leq 1$, we have convergence at $x = -1$ only because the alternating series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ converges for $0 < p \leq 1$; thus, the power series converges on $[-1, 3)$ and diverges outside;

3. when $p \leq 0$, convergence does not occur at the boundary. The power series converges on $(-1, 3)$ and diverges outside.

7.2.2 Elementary Calculus of Taylor Series

Inside the radius of convergence, we shall see that we can differentiate and integrate a power series term per term. But first, we look at simpler properties of Taylor series.

Lemma 7.11. *Let $f : I \to \mathbb{R}$ be an infinitely differentiable function with Taylor series $T^a_\infty f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ about $a \in I$.*

1. *The sum and product of Taylor series about $x = a$ are the Taylor series of the sum and product functions.*
2. *For any $k \in \mathbb{N}$, the Taylor series $T^a_\infty g$ of $g(x) = (x-a)^k f(x)$ about $a \in I$ is given by*
$$T^a_\infty g(x) = (x-a)^k T^a_\infty f(x) = \sum_{n=0}^{\infty} c_n (x-a)^{n+k}.$$
3. *Let $b \in \mathbb{R}$, $k \in \mathbb{N}$, the Maclaurin series for $g(x) = f(bx^k)$ is*
$$T_\infty g(x) = \sum_{n=0}^{\infty} (c_n b^n)\, x^{kn}.$$
4. *The Taylor series of the derivative f' of f is given by the series of term per term derivatives of the Taylor series of f and has the* **same radius of convergence**.

Proof. See Exercise 9 of Sheet 6a for the details.

1. This follows from the similar properties for the Taylor polynomials about $x = a$.
2. This is true by computing the derivatives of $(x-a)^k f(x)$ at $x = a$.
3. This is true by a simple substitution.
4. By definition of $T_\infty f'(x)$,
$$T_\infty f'(x) = \sum_{n=0}^{\infty} \frac{f^{(n+1)}}{n!}(x-a)^n = \sum_{k=0}^{\infty} \frac{d}{dx}\left(\frac{f^{(k)}}{k!}(x-a)^k\right).$$
To get the radius of convergence of $T_\infty f$ we need to evaluate $\rho_2(f) = \lim_{n\to\infty} c_n(f)^{1/n}$ where $c_n(f) = \frac{f^{(n)}(a)}{n!}$. Then
$$\rho_2(f') = \lim_{n\to\infty} n^{1/(n-1)} \left(c_n(f)^{1/n}\right)^{n/(n-1)} = \lim_{n\to\infty} n^{1/(n-1)} \cdot \lim_{n\to\infty} (\rho_2(f))^{n/(n-1)} = \rho_2(f).$$

□

Examples. The Maclaurin series of cosine is $T_\infty \cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$. The Maclaurin series for $f_1(x) = \cos(2x)$, $f_2(x) = x\cos x$ and $f_3(x) = \cos(4x^2)$, respectively, are
$$\sum_{n=0}^{\infty} (-1)^n \frac{2^{2n}}{(2n)!} x^{2n}, \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n+1}, \quad \sum_{n=0}^{\infty} (-1)^n \frac{2^{4n}}{(2n)!} x^{4n}.$$

7.3 Lecture 21: More on Power and Taylor Series

in this lecture we are concerned with the 'inverse' problem for Taylor series. Given a power series, what sort of function does it define?

Definition 7.12. *Let I be an open interval, $a \in I$ and $a \in J \subset I$, an open subinterval of I. Let $g(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ be a power series. If a function $f : I \to \mathbb{R}$ is equal to the power series $f(x) = g(x)$ for $x \in J \subset I$, we say that $g : J \to \mathbb{R}$ is a* **power series expansion** *of f about $x = a$.*

Remark 7.13. *We shall see later that if f has a power series expansion about $x = a$ then the* **power series is unique** *and* **must be its Taylor series** *(7.1), or, in the special case where $a = 0$, its MacLaurin series.*

Example. The series $\sum_{n=0}^{\infty}(x+1)^n = 1 + (x+1) + (x+1)^2 + (x+1)^3 + \dots$ is a power series, a geometric series with common ratio $x + 1$. Providing $|x + 1| < 1$, that is, $x \in (-2, 0)$, it converges to $\frac{1}{|x|}$! In other words the two functions $f : (-2, 0) \to \mathbb{R}$ and $g : (-2, 0) \to \mathbb{R}$ given by

$$f(x) = \frac{1}{|x|} \qquad \text{and} \qquad g(x) = \sum_{n=0}^{\infty}(1+x)^n,$$

are the same, that is $f = g$.

Earlier on we saw that if we started with a known power series, e.g. the geometric series, then if we could compute the sum, we could write down a simpler function to which it is equal. On the other hand, suppose we have a function and want to try to write it as a power series. How do we find the right coefficients of the power series in order to do this?

Suppose we know that $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ but we do not know the numbers $c_0, c_1, \dots c_n, \dots$. Together, the following two theorems help us to compute them.

Theorem 7.14 (Differentiating a power series)**.** *Suppose that $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ is a power series with radius of convergence R. If $|x - a| < R$ then*

$$f'(x) = \sum_{n=0}^{\infty} nc_n(x-a)^{n-1}.$$

Proof. The proof will be sketched in the next lecture. ☐

So given a power series for $f(x)$ we can compute a related power series for $f'(x)$.

Perhaps this theorem looks like it is not saying very much. One way to look at it, is that it says

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big(\sum_{n=0}^{\infty} c_n(x-a)^n\Big) = \sum_{n=0}^{\infty}\Big(\frac{\mathrm{d}}{\mathrm{d}x}c_n(x-a)^n\Big).$$

So we are interchanging the order of summation and differentiation. If a sum is finite, we know from the basic rules of differentiation that, to differentiate the sum, we can differentiate each term separately and add up the answers. So this switch would be valid for a finite sum, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big(\sum_{n=0}^{k} c_n(x-a)^n\Big) = \sum_{n=0}^{k} nc_n(x-a)^{n-1}.$$

However, we have an infinite sum, and infinite sums are much more complicated than finite sums. As we have already seen, results about finite sums e.g commutativity, do not necessarily carry over to infinite sums. So, this theorem is really saying quite a lot.

Both an infinite sum and differentiation are defined in terms of limits. So the theorem is saying that we can switch around the order of these limits without affecting the answer. This sort of thing requires careful justification and proof.

Theorem 7.15. *Suppose that* $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ *is a power series with radius of convergence* $R > 0$. *Then,*

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

Proof. We will prove this by first proving by induction that, providing $|x-a| < R$, we have

$$f^{(k)}(x) = \sum_{n=0}^{\infty} c_{n+k}(n+1)\cdots(n+k)(x-a)^n.$$

Clearly this is true when $k = 0$, because the statement when $k = 0$ is just the definition of f.

Suppose that the result is true whenever $k \leq j$. Then providing $|x-a| < R$, we have

$$f^{(j)}(x) = \sum_{n=0}^{\infty} c_{n+j}(n+1)\cdots(n+j)(x-a)^n.$$

We can apply the previous theorem to the power series $f^{(j)}$ to get

$$f^{(j+1)}(x) = \sum_{n=0}^{\infty} c_{n+j}n(n+1)\cdots(n+j)(x-a)^{n-1}.$$

Hence the result is also true when $k = j+1$. So our claim follows by induction.

So substituting $x = a$ gives

$$f^{(k)}(a) = \sum_{n=0}^{\infty} c_{n+k}(n+1)\cdots(n+k)(a-a)^n = c_k k!.$$

This is true because all of the terms in the sum disappear except for the term when $n = 0$. This is exactly what we want. ∎

Finally, as an immediate corollary, Theorem 7.14 shows that, in the domain $(a-R, a+R)$, a power series is infinitely differentiable and defines a unique function.

Corollary 7.16. *Suppose that two power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ and $\sum_{n=0}^{\infty} b_n(x-x_0)^n$ converge on some interval (x_0-r, x_0+r) to the same function, then $a_n = b_n$, $n \geq 0$, and they are both infinitesimally differentiable.*

Proof. By repeated differentiation,

$$n!a_n = f^{(n)}(x_0) = n!b_n, \quad n \geq 0,$$

and the conclusion. □

7.3.1 The Binomial Theorem

We now turn our attention to an important result we have already seen without proof.

Theorem 7.17 (The Binomial Theorem)**.** *If α is any real number and $|x| < 1$, then*

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-(n-1))}{n!} x^n. \tag{7.6}$$

Remark 7.18. *If α is a positive integer then this just reduces to a special case of the more familiar version of the Binomial Theorem for positive integer powers.*

Theorem 7.19. *If $m \in \mathbb{N}$ then*

$$(1+x)^m = \sum_{n=0}^{m} \binom{n}{m} x^n = \sum_{n=0}^{m} \frac{m(m-1)\cdots(m-(n-1))}{n!} x^n.$$

To see that this is a special case of the previous theorem just notice that if $n-1 \geq m$, that is, $n \geq m+1$, the terms in the series are all zero, so we only have a finite sum. Also, **we do not need the condition on $|x|$ because the sum is finite and so we do not have to worry about convergence.**

Proof. Let $c_n = \dfrac{\alpha(\alpha-1)\cdots(\alpha-(n-1))}{n!} x^n$. Then,

$$R = \lim_{n\to\infty} \frac{c_{n+1}}{c_n} = \frac{(n+1)!\alpha(\alpha-1)\cdots(\alpha-(n-1))}{n!\alpha(\alpha-1)\cdots(\alpha-n)} = \lim_{n\to\infty} \frac{n+1}{n-\alpha} = 1.$$

Hence the series converges if $|x| < 1$ and diverges if $|x| > 1$. We will not worry about showing what happens when $|x| = 1$.

To calculate the convergence we will need to show that c_n corresponds to the general term of the Maclaurin series of $f(x) = (1+x)^\alpha$ and that the remainder tends to 0 for every $|x| < 1$.

It is a straightforward calculation to see that $f^{(n)}(x) = \alpha(\alpha-1)\cdots(\alpha-(n-1))\cdot(1+x)^{\alpha-n}$. Hence, $f^{(n)}(0) = \alpha(\alpha-1)\cdots(\alpha-(n-1))$. And so,

$$R_n f(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\,x^{n+1} = \frac{\alpha(\alpha-1)\cdots(\alpha-n)}{(n+1)!}\cdot(1+\xi)^{(\alpha-n)}\cdot x^n.$$

A close look at this reveals that we can expect trouble if $x < 0$ since then $1+\xi$ can become arbitrarily close to zero and for large n, we'll have $\alpha - n < 0$.

It turns out that a reasonably straightforward proof of convergence can be given if we insist that $0 \leqslant x < 1$ but that the proof requires a lot more effort if $-1 < x \leqslant 0$. We do not give the details here. □

Examples.

1. To find the MacLaurin series for the function $f(x) = \dfrac{1}{\sqrt{9-x}}$ and the values of x for which it converges, we proceed as follows. We have

$$\frac{1}{\sqrt{9-x}} = \frac{1}{\sqrt{9}\sqrt{1-x/9}} = \frac{1}{3}\Big(1-\frac{x}{9}\Big)^{-1/2}.$$

   We can compute this using the Binomial expansion which will result in a valid MacLaurin series expansion if $\left|-\dfrac{x}{9}\right| < 1$, that is, if $-9 < x < 9$. We get

$$\begin{aligned}\frac{1}{3}\Big(1-\frac{x}{9}\Big)^{-1/2} &= \frac{1}{3}\Big(1+\Big(-\frac{1}{2}\Big)\Big(-\frac{x}{9}\Big)+\Big(-\frac{1}{2}\Big)\Big(-\frac{3}{2}\Big)\Big(-\frac{x}{9}\Big)^2+\cdots \\ &\quad + \frac{(-1/2)(-3/2)\cdots(-1/2-(n-1))}{n!}\Big(-\frac{x}{9}\Big)^n+\cdots \\ &= \sum_{n=0}^{\infty}\frac{1\cdot 3\cdot\cdots\cdot(2n-1)}{(18)^n n!}\,x^n = \sum_{n=0}^{\infty}\frac{(2n)!}{(36)^n(n!)^2}\,x^n.\end{aligned}$$

2. To find the MacLaurin series for $(2+3x)^{-\frac{1}{2}}$, we use the Binomial Theorem noting that $p = 1/2$. Rewrite the expression as

$$(2+3x)^{-\frac{1}{2}} = 2^{-\frac{1}{2}}\left(1+\frac{3}{2}x\right)^{-\frac{1}{2}}.$$

   Then, from (7.6):

$$(1+u)^p = 1 + pu + \frac{p(p-1)}{2!}u^2 + \frac{p(p-1)(p-2)}{3!}u^3 + \dots, \quad |u| < 1,$$

   so, on substituting $u = \dfrac{3x}{2}$,

$$\begin{aligned}(2+3x)^{-\frac{1}{2}} &= \frac{1}{\sqrt{2}}\left[1+\frac{\left(-\frac{1}{2}\right)}{1}\left(\frac{3}{2}x\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{1\cdot 2}\left(\frac{3}{2}x\right)^2+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{1\cdot 2\cdot 3}\left(\frac{3}{2}x\right)^3+\dots\right] \\ &= \frac{1}{\sqrt{2}}\left(1-\frac{3}{4}x+\frac{27}{32}x^2-\frac{135}{128}x^3+\dots\right)\end{aligned}$$

   which is convergent for $\left|\frac{3}{2}x\right| < 1$ that is $|x| < \frac{2}{3}$.

7.4 Extra curricular material: General Theorem About Taylor Series

In this lecture we collect together the information we have on power series and add some additional properties we are going to prove next term in Analysis 2 (MA2731).

Theorem 7.20 (Power and Taylor's series)**.** *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers such that $\rho = \lim_{n\to\infty} |a_n|^{1/n}$ is defined. When $\rho > 0$, let $R = \dfrac{1}{\rho}$ and, when $\rho = 0$, let $R = \infty$. The power series*

$$f(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k$$

1. *defines the values $f(x)$ of a function $f : (x_0 - R, x_0 + R) \to \mathbb{R}$.*
2. *The function f is infinitely differentiable (in particular continuous) for $x \in (x_0 - R, x_0 + R)$.*
3. *The values $f(x)$ are not defined for $|x - x_0| > R$. What happens to $x = x_0 \pm R$ needs to be investigated individually.*
4. *For any $n \in \mathbb{N}$, the derivatives $f^{(n)}$ of f have also a representation as power series with the* **same** *radius of convergence R. The coefficients of the power series of the n-th derivative are obtained by differentiating n-times each of the terms of the power series for f. The function f has the Taylor's series representation*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n, \quad x \in (x_0 - R, x_0 + R).$$

5. *Let $x_0 - R < a < b < x_0 + R$, the integral*

$$\int_a^b f(x)\, dx$$

of f on $[a, b]$ has a representation in an absolutely convergent series obtained by integrating each term of the power series defining f.

7.4.1 Examples of Power Series Revisited

Now we derive consequences of Theorem 7.20.

1. The geometric series is

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{k=0}^{\infty} x^k, \quad |x| < 1.$$

for $x_0 = 0$ and $R = 1$. Hence we can differentiate term-by-term and we can integrate term-by-term to get the following series

$$\begin{aligned} \frac{1}{(1-x)^2} &= 1 + 2x + 3x^2 + \cdots = \sum_{k=1}^{\infty} kx^{k-1}, \quad |x| < 1, \\ -\ln(1-x) &= \int_0^x \frac{dt}{1-t} = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots = \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1}, \quad |x| < 1. \end{aligned}$$

The results that have been covered say nothing about the convergence or divergence at any point x with $|x| = 1$. In the case of the series for $1/(1-x)^2$ the terms do not tend to 0 when $|x| = 1$ and the series diverges for all $|x| = 1$. In the case of the series for $-\ln(1-x)$, the series is the harmonic series when $x = 1$ and hence diverges when $x = 1$. Notice also that

$$\lim_{x \to 1^-} -\ln(1-x) = \infty.$$

When $x = -1$, the left hand side is $-\ln(2)$ and the right hand side is the alternating harmonic series. We shall see in the next subsection that the series converges so

$$\ln(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots .$$

2. The geometric series

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots ,$$

converges for $|x| < 1$. Now from $x = \tan y$, $y = \arctan x$ and

$$\frac{dx}{dy} = \sec^2 y = 1 + \tan^2 y = 1 + x^2,$$

we get

$$\frac{dy}{dx} = \frac{1}{1+x^2}.$$

We can integrate term-by-term to obtain

$$\arctan x = \int_0^x \frac{dt}{1+t^2} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots .$$

Note that arctan is defined for all $x \in \mathbb{R}$, but this representation as a power series is only true for $|x| < 1$.

3. Theorem 7.20 does not make any claims about the convergence of the power series, or its derivatives, at $|x - x_0| = R$. Indeed, anything can happen,

   (a) no convergence at all,

   (b) convergence at one point, or

   (c) convergence at both end points.

   Also, **the power series may converge, but not its derivative** etc. For instance, the series
$$\sum_{n=1}^{\infty} \frac{x^n}{n^2}$$
   converges at both end-points $x = \pm 1$, but its derivative converges (conditionally) only at $x = -1$.

7.4.2 Leibniz' Formulas For $\ln 2$ and $\pi/4$

Leibniz[1] showed that
$$\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots = \ln 2$$
and
$$\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots = \frac{\pi}{4}$$
Both formulas arise by setting $x = 1$ in the Maclaurin series for
$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} - \cdots$$
$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} - \cdots$$

This is only justified if you show that the series actually converge at $x = 1$, which we'll do here for the first of these two formulas. The proof for the second formula is similar. The following is not Leibniz' original proof. Begin with the geometric sum
$$1 - x + x^2 - x^3 + \cdots + (-1)^n x^n = \frac{1}{1+x} + \frac{(-1)^{n+1} x^{n+1}}{1+x}$$
Then you integrate both sides from $x = 0$ to $x = 1$ and get
$$\begin{aligned} \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \cdots + (-1)^n \frac{1}{n+1} &= \int_0^1 \frac{dx}{1+x} + (-1)^{n+1} \int_0^1 \frac{x^{n+1}\,dx}{1+x} \\ &= \ln 2 + (-1)^{n+1} \int_0^1 \frac{x^{n+1}\,dx}{1+x} \end{aligned}$$

[1]G.W.L. Leibniz (1646-1716) was a German mathematician and philosopher who contributed to many different areas of human knowledge. He is famous for his version of calculus and the dispute with Newton. Many of the notation and ideas where made explicit by Leibnitz. His formalism was to prove vital in latter development. He used the notation $\int$, gave the product rule for differentiation and the derivative of x^r for integral and fractional r, but he never thought of the derivative as a limit. Another great achievements in mathematics were the development of the binary system of arithmetic, his work on determinants which arose from his developing methods to solve systems of linear equations.

Instead of computing the last integral, you estimate it by noting that

$$0 \le \frac{x^{n+1}}{1+x} \le x^{n+1}$$

implying that

$$0 \le \int_0^1 \frac{x^{n+1}\,dx}{1+x} \le \int_0^1 x^{n+1}\,dx = \frac{1}{n+2}.$$

Hence,

$$\lim_{n\to\infty} (-1)^{n+1} \int_0^1 \frac{x^{n+1}\,dx}{1+x} = (-1)^{n+1} \lim_{n\to\infty} \int_0^1 \frac{x^{n+1}\,dx}{1+x} = 0,$$

and we get

$$\lim_{n\to\infty} \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \cdots + (-1)^n \frac{1}{n+1} = \ln 2 + \lim_{n\to\infty} (-1)^{n+1} \int_0^1 \frac{x^{n+1}\,dx}{1+x} = \ln 2.$$

As a final example, we can try to construct the sum of the series

$$\sum_{k=1}^{\infty} kx^k,$$

using directly Theorem 7.20 by manipulating the series to obtain derivatives or integrals of known series. The radius of convergence is $R = 1$ and $x_0 = 0$. For $|x| < 1$, we write

$$\begin{aligned}
\sum_{k=1}^{\infty} kx^k &= \sum_{k=1}^{\infty}(k+1)x^k - \sum_{k=1}^{\infty} x^k = \sum_{k=1}^{\infty} \left(x^{k+1}\right)' - \frac{x}{x-1} \\
&= \left(x \sum_{k=1}^{\infty} x^k\right)' - \frac{x}{x-1} = \left(\frac{x^2}{1-x}\right)' - \frac{x}{x-1} = \frac{x}{(1-x)^2}.
\end{aligned}$$

7.5 Extra curricular material: Taylor Series and Fibonacci Numbers

In this lecture we shall look at examples of use of Taylor/Maclaurin series in different areas of Mathematics.

7.5.1 Taylor Series in Number Theory

First, an interesting application of Taylor's Theorem and the integral form of the remainder.

Theorem 7.21. *The number e is irrational.*

Proof. Recall that e is irrational means that e cannot be written as a fraction of two integers. Let $f(x) = e^x$. We know that

$$e^x = \sum_{k=0}^{n} \frac{1}{k!} + R_n f(x).$$

Take $x > 0$. Then, the revisited Taylor's Theorem 6.6 tells us that the **integral form** of $R_n f$ is

$$\begin{aligned} R_n f(x) &= \int_0^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n \, dt = \int_0^x \frac{e^t}{n!}(x-t)^n \, dt \leq \frac{e^x}{n!} \int_0^x (x-t)^n \, dt \\ &= \frac{e^x}{n!}\Big[-\frac{(x-t)^{n+1}}{n+1}\Big]_0^x = \frac{e^x x^{n+1}}{(n+1)!}. \end{aligned}$$

Now take $x = 1$ and an integer $N > e$. Then,

$$e = \sum_{k=0}^{n} \frac{1}{k!} + R_n f(1),$$

where

$$0 < R_n f(1) < \frac{N}{(n+1)!}.$$

Suppose, for contradiction, that e is rational. Let $e = \dfrac{p}{q}$ where p and q are positive integers. Choose n so that $n > q$ and $n > 3$. Then,

$$\frac{p}{q} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + R_n f(1).$$

So,

$$\frac{pn!}{q} = n! + n! + \frac{n!}{2!} + \frac{n!}{3!} + \cdots + 1 + n! R_n f(1).$$

Every term in this equation other than possibly $n!R_n f(1)$ is an integer, so $n!R_n f(1)$ must be an integer. However,

$$0 < n! R_n f(1) < \frac{N}{n+1} < 1.$$

So, $n!R_n f(1)$ is not an integer after all, giving a contradiction. □

7.5.2 Taylor's Formula and Fibonacci Numbers

Now, consider the function rule

$$f(x) = \frac{1}{1-x-x^2}.$$

How can we calculate $T_\infty f$? Using the definition seems complicated because the derivatives of f will involve complicated rational functions:

$$f'(x) = \frac{1+2x}{(1-x-x^2)^2}, \quad f''(x) = \frac{2(2+3x+3x^2)}{(1-x-x^2)^3} \dots$$

What else could we do? An idea is the following. Let

$$T_\infty f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

be its Taylor series. Considering the Taylor polynomials $T_n f$ for large n, due to Lagrange's Remainder Theorem, we have, for any n,

$$\frac{1}{1-x-x^2} = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_n x^n + R_n f(x).$$

Multiply both sides with $1-x-x^2$, ignoring the terms of degree larger than n, you get

$$\begin{aligned}
1 &= (1-x-x^2)\cdot(c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n) \\
&= \begin{array}{ccccccccc} c_0 & + & c_1 x & + & c_2 x^2 & + & \cdots & + & c_n x^n \\ & - & c_0 x & - & c_1 x^2 & - & \cdots & - & c_{n-1} x^n \\ & & & - & c_0 x^2 & - & \cdots & - & c_{n-2} x^n \end{array} \\
&= c_0 + (c_1 - c_0)x + (c_2 - c_1 - c_0)x^2 + \cdots + (c_n - c_{n-1} - c_{n-2})x^n.
\end{aligned}$$

Compare the coefficients of powers x^k on both sides for $k = 0, 1, \ldots, n$ and you find

$$c_0 = 1, \quad c_1 - c_0 = 0, \quad c_2 - c_1 - c_0 = 0, \quad \cdots, \quad c_n - c_{n-1} - c_{n-2} = 0.$$

Therefore the coefficients of the Taylor series $T_\infty f(x)$ satisfy the **(second order) recursion relation**

$$c_n = c_{n-1} + c_{n-2},\ n \geq 2, \tag{7.7}$$

with initial data

$$c_0 = 1 = c_1 = 1. \tag{7.8}$$

The numbers satisfying (7.7) and (7.8) are known in other aspects mathematics, they are the **Fibonacci numbers F_n** with $F_0 = F_1 = 1$. We can calculate them:

$$\begin{aligned}
F_2 &= F_1 + F_0 = 1 + 1 = 2, \\
F_3 &= F_2 + F_1 = 2 + 1 = 3, \\
F_4 &= F_3 + F_2 = 3 + 2 = 5,
\end{aligned}$$

etc.

The first time they appeared was in the model problem of calculating the growth in a population of 'rabbits' such that each pair has exactly one pair of offsprings each year but cannot reproduce for one year (as juveniles). Since, they appeared in other area of biology (in a more realistic way).

Since it is much easier to compute the Fibonacci numbers one by one than it is to compute the derivatives of $f(x) = 1/(1-x-x^2)$, this is a better way to compute the Taylor series of $f(x)$ than just directly from the definition. **The Fibonacci numbers are encoded into the Taylor series of** $f(x) = 1/(1-x-x^2)$. This appears in area of higher mathematics, for instance in Probability and Statistics. The term is to say that f is the **generating function** for the Fibonacci numbers.

7.5.3 More about the Fibonacci Numbers

In this section we are going to see a way to compute the Taylor series of any **any rational function**. As a consequence we shall be able to find an explicit formula for the Fibonacci numbers (7.7).

You already know the trick: find the partial fraction decomposition of the given rational function. Ignoring the case that you have quadratic expressions in the denominator, this lets you represent your rational function as a sum of terms of the form

$$\frac{A}{(x-a)^p}.$$

These are easy to differentiate any number of times, and thus they allow you to write their Taylor series. If we apply this idea to $f(x) = 1/(1-x-x^2)$, from the previous example 7.5.2, we get the following.

First, factor the denominator. We have

$$1 - x - x^2 = -(x^2 + x - 1) = -(x - x_-)(x - x_+)$$

where $x_\pm$ are the roots of the quadratic polynomial

$$x^2 + x - 1 = 0.$$

Using the general formula for the roots of a quadratic equation, we get,

$$x_\pm = \frac{-1 \pm \sqrt{5}}{2}.$$

Define the **golden ratio** ϕ as

$$\phi = \frac{1+\sqrt{5}}{2} \approx 1.618\,033\,988\,749\,89\ldots.$$

Note that

$$\frac{1}{\phi} = \frac{2}{\sqrt{5}+1} = \frac{2(\sqrt{5}-1)}{5-1} = \frac{\sqrt{5}-1}{2}.$$

And so, $\phi + \frac{1}{\phi} = \sqrt{5}$ and $\phi - \frac{1}{\phi} = 1$. Then,

$$x_- = -\phi, \quad x_+ = \frac{1}{\phi}.$$

So,

$$1 - x - x^2 = -(x - \frac{1}{\phi})(x + \phi).$$

Hence, $f(x)$ can be written as

$$f(x) = \frac{1}{1 - x - x^2} = \frac{-1}{(x - \frac{1}{\phi})(x + \phi)} = \frac{A}{x - \frac{1}{\phi}} + \frac{B}{x + \phi},$$

where

$$A = \frac{-1}{\frac{1}{\phi} + \phi} = \frac{-1}{\sqrt{5}}, \qquad B = \frac{1}{\frac{1}{\phi} + \phi} = \frac{1}{\sqrt{5}}.$$

Therefore,

$$f(x) = \frac{1}{1 - x - x^2} = \frac{\phi}{\sqrt{5}(1 - x\phi)} + \frac{1/\phi}{\sqrt{5}(1 + \frac{x}{\phi})}.$$

Using the Binomial Theorem, we find the expansions of $(1 - x\phi)^{-1}$ and $(1 + \frac{x}{\phi})^{-1}$ as

$$\begin{aligned} \frac{1}{1 - x\phi} &= 1 + (x\phi) + (x\phi)^2 + \cdots + (x\phi)^n + \cdots \\ \frac{1}{1 + \frac{x}{\phi}} &= 1 - \frac{x}{\phi} + (\frac{x}{\phi})^2 + \cdots + (-1)^n(\frac{x}{\phi})^n + \cdots \end{aligned}$$

Collecting the terms together, we find

$$F_n = \frac{1}{\sqrt{5}} \left(\phi^{n+1} + \frac{(-1)^n}{\phi^{n+1}} \right). \tag{7.9}$$

Note. We can check that (7.9) gives F_0 and F_1 correctly. For that, recall that $\phi^2 = 1 + \phi$. And so,

$$\begin{aligned} F_0 &= \frac{1}{\sqrt{5}} \left(\phi + \frac{1}{\phi} \right) = \frac{\sqrt{5}}{\sqrt{5}} = 1, \\ F_1 &= \frac{1}{\sqrt{5}} \left(\phi^2 - \frac{1}{\phi^2} \right) = \frac{\phi}{1 + \phi^2} \left(\frac{\phi^4 - 1}{\phi^2} \right) = \frac{\phi^2 - 1}{\phi} = 1. \end{aligned}$$

Remark 7.22. *As a final remark, there are some problems where the initial numbers are $f_0 = 0$ and $f_1 = 1$. In that case we can see that $f_n = F_{n-1}$, $n \geq 1$, and their generating function is now* $\frac{x}{1 - x - x^2}$.

7.6 Extra curricular Christmas treat: Series of Functions Can Be Difficult Objects

In this appendix we look at two phenomena illustrating difficult behaviour what can happen with series of functions.

1. In the first example, we show that a Taylor series for a function f about $x = a$ may converge everywhere but **never** to $f(x)$ outside of $x = a$.
2. In a second example, we show that **Fourier series** can converge to bizarre function. We show an explicit example where the limit is a continuous function, but nowhere differentiable, hence impossible to draw.

7.6.1 What Can Go 'Wrong' with Taylor Approximation?

Let f be the function

$$f(x) = \begin{cases} e^{-1/x^2}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

We are going to show that the Maclaurin series of f is $T_\infty f(x) = 0$ even if $f(x) > 0$ for $x \neq 0$. Therefore, **even if a Taylor series of some function f converges, you cannot be sure that it converges to f, it could converge to a different function**.

Proposition 7.23. *The Maclaurin series of f is $T_\infty f(x) = 0$.*

Proof. We are going to show that all derivatives of f are 0 at the origin. We proceed in a few steps.

1. We need to prove that for all $n \in \mathbb{N}$, providing $x \neq 0$,

$$f^{(n)}(x) = \frac{p(x)e^{-1/x^2}}{x^k}$$

where $k \in \mathbb{N}$ and p is a polynomial, both depending on n. We will prove this by induction on n.

For $x \neq 0$, we have $f^{(1)}(x) = \dfrac{2e^{-1/x^2}}{x^3}$, so the statement is true when $n = 1$.

Suppose the statement is true when $n = m$, that is for all $x \neq 0$, $f^{(m)}(x) = \dfrac{p(x)\, e^{-1/x^2}}{x^k}$ for some $k \in \mathbb{N}$ and polynomial p. Then, providing $x \neq 0$,

$$\begin{aligned} f^{(m+1)}(x) = \frac{\mathrm{d}}{\mathrm{d}x} f^{(m)}(x) &= \frac{p'(x)e^{-1/x^2}x^k + 2p(x)/x^3 \cdot x^k e^{-1/x^2} - p(x)e^{-1/x^2}kx^{k-1}}{x^{2k}} \\ &= \frac{(x^3p'(x) + 2p(x) - kx^2p(x))e^{-1/x^2}}{x^{k+3}}, \end{aligned}$$

which is of the required form. Hence the statement follows by induction.

2. Now show that $f^{(n)}(0) = 0$ using the rigorous definition of a derivative. To evaluate the limit, you will need to find $\lim_{x\to 0} \frac{e^{-1/x^2}}{x^k}$ where $k \in \mathbb{N}$. This can be done by taking logs and then using L'Hôpital's Rule.

   We prove this by induction too. This time we will show that for all $n \geq 0$, $f^{(n)}(0) = 0$. This is clearly true for $n = 0$. Suppose it is true for $n = m$. Then

$$f^{(m+1)}(0) = \lim_{x\to 0} \frac{f^{(m)}(x)}{x} = \lim_{x\to 0} \frac{p(x)e^{-1/x^2}}{x^{k+1}} = \lim_{x\to 0} p(x) \lim_{x\to 0} \frac{e^{-1/x^2}}{x^{k+1}}.$$

   The first limit is finite and we will show that the second limit is 0 using L'Hôpital's Rule. This will show that the limit we want is equal to zero and establish the result by induction.

   It will be enough to show that $\lim_{x\to 0} \frac{e^{-1/x^2}}{|x|^{k+1}} = 0$.

$$\ln\left(\frac{e^{-1/x^2}}{|x|^{k+1}}\right) = -\frac{1}{x^2} - (k+1)\ln|x| = -\frac{1 + (k+1)x^2 \ln|x|}{x^2}.$$

   We will show that $\lim_{x\to 0} x^2 \ln|x| = 0$. This can be done with a standard application of L'Hôpital's rule by writing

$$x^2 \ln|x| = \frac{\ln|x|}{1/x^2}.$$

   Therefore $\lim_{x\to 0}(1 + (k+1)x^2 \ln|x|) = 1$ and so

$$\lim_{x\to 0}\left(-\frac{1}{x^2} - (k+1)\ln|x|\right) = -\lim_{x\to 0} \frac{1 + (k+1)x^2 \ln|x|}{x^2} = -\infty.$$

   Because exp is continuous, if $g(x) \to -\infty$ as $x \to 0$ then $\exp(g(x)) \to 0$ as $x \to 0$ so

$$\lim_{x\to 0} \frac{e^{-1/x^2}}{|x|^{k+1}} \lim_{x\to 0} \exp\left(-\frac{1}{x^2} - (k+1)\ln|x|\right) = 0.$$

3. We are told that $f(0) = 0$ and have proved that $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$. So, all the coefficients of the MacLaurin series are 0. Consequently the MacLaurin series is identically zero.

□

7.6.2 The Day That All Chemistry Stood Still

The rate at which a chemical reaction '$A \to B$' proceeds depends among other things on the temperature at which the reaction is taking place. This dependence is described by the **Arrhenius law** which states that the rate at which a reaction takes place is proportional to

$$A(T) = e^{-\frac{\Delta E}{kT}}$$

where ΔE is the amount of energy involved in each reaction, k is Boltzmann's constant and T is the temperature in degrees Kelvin. If you ignore the constants ΔE and k (i.e. if you set them equal to one by choosing the right units) then the reaction rate is proportional to

$$f(T) = e^{-1/T}.$$

If you have to deal with reactions at low temperatures you might be inclined to replace this function with its Taylor series at $T = 0$, or at least the first non-zero term in this series. If you were to do this, you would be in for a surprise. To see what happens, look at the following function,

$$f(x) = \begin{cases} e^{-1/x}, & x > 0, \\ 0, & x \leqslant 0. \end{cases}$$

This function goes to zero **very** quickly as $x \to 0$. In fact one has, setting $t = 1/x$,

$$\lim_{x\to 0} \frac{f(x)}{x^n} = \lim_{x\to 0} \frac{e^{-1/x}}{x^n} = \lim_{t\to\infty} t^n e^{-t} = 0.$$

As $x \to 0$, this function vanishes faster than any power of x.

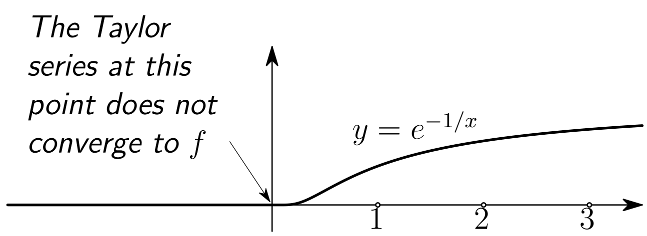


Figure 7.1: An innocent looking function with an unexpected Taylor series.

If you try to compute the Taylor series of f you need its derivatives at $x = 0$ of all orders. These can be computed (see last weeks example), and the result turns out to be that **all derivatives of f vanish at $x = 0$,**

$$f(0) = f'(0) = f''(0) = f^{(3)}(0) = \cdots = 0.$$

The Taylor series of f is therefore

$$T_\infty f(x) = 0 + 0 \cdot x + 0 \cdot \frac{x^2}{2!} + 0 \cdot \frac{x^3}{3!} + \cdots = 0.$$

Clearly this series converges (all terms are zero, after all), but instead of converging to the function f we started with, it converges to the function $g = 0$.

What does this mean for the chemical reaction rates and Arrhenius' law? We wanted to 'simplify' the Arrhenius law by computing the Taylor series of $f(T)$ at $T = 0$, but we have just seen that all terms in this series are zero. Therefore replacing the Arrhenius reaction rate by its Taylor series at $T = 0$ has the effect of setting all reaction rates equal to zero.

7.6.3 Series Can Define Bizarre Functions: Continuous but Nowhere Differentiable Functions

As a final remark about series, Karl Weierstrass in 1872 considered the function

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \sum_{j=0}^{\infty} f_j(x), \quad \text{where } f_j(x) = \frac{1}{2^j} \cos 3^j x. \tag{7.10}$$

For all x, $|f_j(x)| \leqslant 1/2^j$ and, since the geometric series $\sum_{j=0}^{\infty} 2^{-j}$ converges, the series converges uniformly on $\mathbb{R}$ and consequently defines a continuous function. The particularly interesting property of f that Weierstrass proved (which will not be shown here) is that f is not differentiable at any point. The function f is an example of a continuous but nowhere differentiable function.

In Figure 7.2 we represent the finite sums $n - 6 + \sum_{j=1}^{n} \frac{1}{2^j} \cos 3^j x$ that are translations upwards (by an amount of $n-3$) of the graphs so that we can compare them. Note that as n increases the graphs become more and more jaggy.

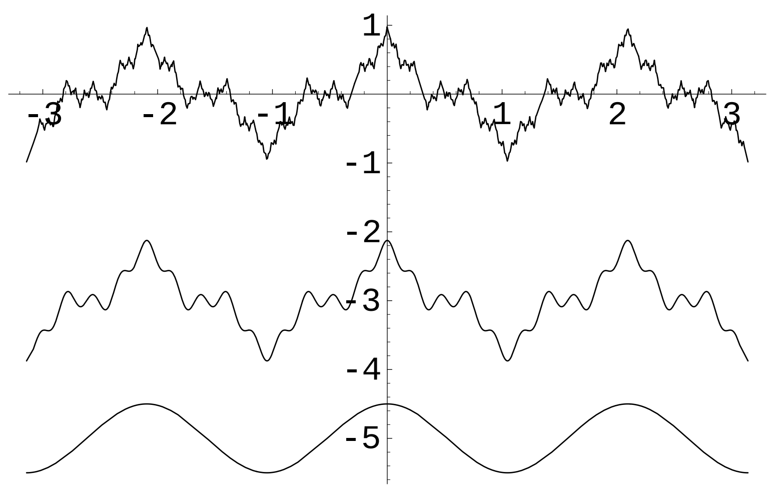


Figure 7.2: Example (7.10) of a continuous nowhere differentiable real function

Summary of Chapter 6

We have seen:

- To be able to calculate the radius of convergence of a power series and to evaluate the convergence at the boundary of the convergence interval.

7.7 Exercise Sheet 6

7.7.1 Exercise Sheet 6a

1. Find the MacLaurin series of the following function rules:

   (a) $\sqrt[3]{1 + 2t + t^2}$.

   (b) $e^{-t} \sin 2t$.

   (c) $\arcsin t$

   (d) $\dfrac{1}{\sqrt{1 - t^2}}$.

   (e) $f(x) = \begin{cases} \dfrac{\sin(x)}{x}, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0. \end{cases}$

   (f) $\dfrac{\ln(1 + x)}{x}$.

   (g) $e^{-t} \cos t$.

2. Determine the values of x for which the following function rules has a valid power series expansion about 0.

   (a) $\cos x$,

   (b) $\sin x$,

   (c) $\ln(1 + x)$.

3. Determine the radius of convergence of the following power series.

   (a) $\sum_{k=0}^{\infty} \dfrac{k^3 x^k}{3^k}$.

   (b) $\sum_{k=1}^{\infty} k^k x^k$.

   (c) $\sum_{k=0}^{\infty} \dfrac{2^k x^k}{k!}$.

   (d) $\sum_{k=0}^{\infty} k! x^k$.

   (e) $\sum_{k=0}^{\infty} \left(\dfrac{k^2}{5^k}\right) x^k$.

   (f) $\sum_{k=1}^{\infty} \left(\dfrac{k}{2^k}\right) x^k$.

   (g) $\sum_{k=1}^{\infty} \left(\dfrac{1}{k^k}\right) x^k$.

4. Finish the proof of Proposition 7.4. Show that the sequence $\{\dfrac{x^{n+1}}{(n+1)!}\}_{n=1}^{\infty}$ has limit 0.

5. Determine the **exact sums** of the power series in Exercises 3(i), 3(iii), 3(v) and 3(vi).

6. Use Maclaurin series to find an expression of the integral $\displaystyle\int_0^1 e^{-x^2}\, dx$ as a series.

7. Let $\sum_{n=0}^{\infty} c_n(x-a)^n$ be a power series. Prove that, when they exist,

$$\rho_1 = \lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right|, \quad \text{and} \quad \rho_2 = \lim_{n\to\infty} |c_n|^{1/n},$$

are equal.

8. Complete the proof of Theorem 7.10.

9. Complete the proof of Lemma 7.11.

10. Show that we cannot apply the Ratio Test to the power series $\sum_{n=1}^{\infty} a_n x^n$ where $a_n = 1$ if n is odd, $a_n = 2$ when n is even, but we can apply the Root Test. Find its radius of convergence.

Short Feedback for Exercise Sheet 6a

1. (a) Hint: use the Binomial Formula.
   (b) Hint: Use de Moivre's Formula (see Level 1)
   $$e^{ix} = \cos x + i \sin x. \tag{7.11}$$
   (c) Hint: Differentiate $\arcsin t$.
   (d) Hint: use the Binomial Formula.
   (e) Hint: use the calculus of Taylor series.
   (f) Hint: use the calculus of Taylor series.
   (g) Hint: Use (7.11).

2. (a) $x \in \mathbb{R}$,
   (b) $x \in \mathbb{R}$,
   (c) $x \in (-1, 1)$.

3. (a) $R = 3$.
   (b) $R = 0$.
   (c) $R = \infty$.
   (d) $R = 0$.
   (e) $R = 5$.
   (f) $R = 2$.
   (g) $R = \infty$.

4. Use the convergence of a monotone sequence to establish the existence of a limit, then calculate it.

5. 3(i) You may need $\sum_{k=1}^{\infty} ky^k = \frac{y}{(1-y)^2}$.

3(iii) e^{2x}.

3(v) You may need to calculate $\sum_{k=1}^{\infty} k^2 y^k$.

3(vi) $\frac{2x}{(2-x)^2}$.

6. $1 - \frac{1}{3} + \frac{1}{5 \cdot 2} - \frac{1}{7 \cdot 3!} + \frac{1}{9 \cdot 4!} - \dots$.

7.

8.

9.

10. $R = 1$.

7.7.2 Feedback for Exercise Sheet 6a

1. (a) $\sqrt[3]{1+2t+t^2} = \sqrt[3]{(1+t)^2} = (1+t)^{2/3}$. The answer follows from Newton's binomial formula. The Maclaurin series of $(1+t)^{2/3}$ is

$$\sum_{n=0}^{\infty} \binom{2/3}{n} t^n.$$

(b) Let $f(t) = e^{-t}\sin(2t)$. To simplify calculation we are going to use a 'neat trick': to write $\sin(2t)$ as a complex exponential:

$$\sin(2t) = \frac{e^{2it} - e^{-2it}}{2i} = \frac{i}{2}\left(e^{-2it} - e^{2it}\right).$$

Combining with e^{-t}, we find

$$f(t) = \frac{i}{2}\left(e^{-(1+2i)t} - e^{(-1+2i)t}\right).$$

Therefore we can use the Maclaurin expansion for the exponential:

$$T_\infty f = \frac{i}{2}\sum_{n=0}^{\infty}(-1)^n(1+2i)^n\frac{t^n}{n!} - \frac{i}{2}\sum_{n=0}^{\infty}(-1+2i)^n\frac{t^n}{n!}.$$

To be able to calculate in a straightforward manner the powers of the complex numbers, we write them in polar form

$$1+2i = \sqrt{5}\,e^{i\alpha}, \quad -1+2i = -\sqrt{5}\,e^{-i\alpha}$$

where $\tan\alpha = 2$. We can now go back to $T_\infty f$:

$$\begin{aligned}
T_\infty f(x) &= \frac{i}{2}\sum_{n=0}^{\infty}(-1)^n\left(\sqrt{5}\,e^{i\alpha}\right)^n\frac{t^n}{n!} - \frac{i}{2}\sum_{n=0}^{\infty}(-1)^n\left(-\sqrt{5}\,e^{-i\alpha}\right)^n\frac{t^n}{n!} \\
&= \frac{i}{2}\sum_{n=0}^{\infty}\left((-1)^n\frac{5^{n/2}}{n!}e^{in\alpha}t^n - (-1)^n\frac{5^{n/2}}{n!}e^{-in\alpha}t^n\right) \\
&= \sum_{n=0}^{\infty}(-1)^n\frac{5^{n/2}}{n!}\frac{i}{2}\left(e^{in\alpha} - e^{-in\alpha}\right)t^n \\
&= \sum_{n=0}^{\infty}(-1)^n\frac{5^{n/2}}{n!}\sin(n\alpha)\,t^n.
\end{aligned}$$

(c) The derivative of $\arcsin t$ is $\dfrac{1}{\sqrt{1-t^2}}$. So you need to integrate the result (7.12) of the next exercise. We find

$$T_\infty \arcsin(t) = \int\left(\sum_{n=0}^{\infty}\frac{(2n)!}{4^n(n!)^2}t^{2n}\right) = \sum_{n=0}^{\infty}\frac{(2n)!}{4^n(n!)^2}\frac{t^{2n+1}}{2n+1}.$$

(d) We use the Binomial Theorem or direct differentiation on $g(z) = (1-z)^{-1/2}$, then replace z by t^2. We find that the Maclaurin series expansion of g is

$$\sum_{n=0}^{\infty} \frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots 2n} z^n = \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (n!)^2} z^n$$

multiplying top and bottom by the product of the first n even numbers:

$$2 \cdot 4 \cdots 2n = 2^n (n!).$$

Therefore, the Maclaurin series we seek is

$$\sum_{n=0}^{\infty} \frac{(2n)!}{4^n (n!)^2} t^{2n}. \tag{7.12}$$

(e) Because $T_\infty \sin(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n+1}}{2n+1!}$, we have

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n}}{2n+1!}.$$

(f) Recall that

$$(\ln(1+x))' = \frac{1}{1+x}.$$

Because the Maclaurin expansion of $\frac{1}{1+x}$ is $\sum_{n=0}^{\infty} (-1)^n x^n$, integrating, we get the Maclaurin expansion of $\ln(1+x)$ as

$$\sum_{n=0}^{\infty} (-1)^n \int x^n \, dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}.$$

Therefore we obtain

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n+1}.$$

(g) We can differentiate $f(t) = e^{-t} \cos t$. We get

$$\begin{aligned} f'(t) &= -e^{-t}(\cos t + \sin t), \\ f''(t) &= -2e^{-t} \sin t, \\ f'''(t) &= 2e^{-t}(\cos t - \sin t), \\ f^{(4)}(x) &= -4f(x). \end{aligned}$$

And so $f(0) = 1$, $f'(0) = -1$, $f''(0) = 0$, $f'''(0) = 2$ and $f^{(4)}(0) = -4f(0) = -4$, hence, for $n \geq 0$,

$$\begin{aligned} f^{4n}(0) &= (-4)^n, \\ f^{4n+1}(0) &= -(-4)^n, \\ f^{4n+2}(0) &= 0, \\ f^{4n+3}(0) &= 2(-4)^n. \end{aligned}$$

So the Maclaurin series expansion is

$$
\begin{aligned}
T_\infty f(t) &= \sum_{n=0}^{\infty} f^{4n}(0)\frac{t^{4n}}{4n!} + f^{4n+1}(0)\frac{t^{4n+1}}{(4n+1)!} + f^{4n+3}(0)\frac{t^{4n+3}}{(4n+3)!} \\
&= \sum_{n=0}^{\infty}\left((-4)^n\frac{t^{4n}}{4n!} - (-4)^n\frac{t^{4n+1}}{(4n+1)!} + 2(-4)^n\frac{t^{4n+3}}{(4n+3)!}\right) \\
&= \sum_{n=0}^{\infty}(-4)^n\frac{t^{4n}}{4n!}\left(1 - \frac{t}{4n+1} + \frac{2t^3}{(4n+1)(4n+2)(4n+3)}\right) \\
&= \sum_{n=0}^{\infty}(-4)^n\frac{t^{4n}}{4n+1!}\left(4n+1-t+\frac{t^3}{(2n+1)(4n+3)}\right).
\end{aligned}
$$

2. (a) The remainder term $R_n(x)$ is equal to $\frac{f^{(n)}(\zeta_n)}{n!}x^n$ for some ζ_n. For either the cosine or sine and any n and ζ, we have $|f^{(n)}(\zeta)| \le 1$. So, $|R_n(x)| \le \frac{|x|^n}{n!}$. But we know $\lim_{n\to\infty}\frac{|x|^n}{n!} = 0$ and hence $\lim_{n\to\infty} R_n(x) = 0$.

(b) We know that the MacLaurin expansion for $\sin x$ is $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}$. To test when this converges to $\sin x$, we need to test if the remainder $|R_{n,0}(x)|$ converges to 0. We will use the Sandwich Theorem to show this. From Taylor's Theorem we know that

$$|R_{n,0}(x)| \le \left|\frac{M}{(n+1)!}x^{n+1}\right|,$$

where M is the absolute maximum of $|f^{(n+1)}(t)|$ on the interval $[0,x]$ or $[x,0]$ (depending on the sign of x). But all the derivatives of $\sin x$ are either $\pm\sin x$ or $\pm\cos x$ and we have $|\sin x| \le 1$ and $|\cos x| \le 1$ for all x. Hence, $M \le 1$ and consequently

$$|R_{n,0}(x)| \le \frac{1}{(n+1)!}|x|^{n+1}.$$

From Exercise 4 we know that $\displaystyle\lim_{n\to\infty}\frac{1}{(n+1)!}|x|^{n+1} = 0$. Consequently, by the Sandwich Theorem, $\displaystyle\lim_{n\to\infty}|R_{n,0}(x)| = 0$.

(c) We know that if $f(x) = \ln(1+x)$ has a power series expansion $\ln(1+x) = \displaystyle\sum_{n=0}^{\infty} c_n x^n$ then $c_n = \dfrac{f^{(n)}(0)}{n!}$. We know that $f^{(n)}(x) = \dfrac{(-1)^{n-1}(n-1)!}{(1+x)^n}$ and so $f^{(n)}(0) = (-1)^{n-1}(n-1)!$.

Hence, if $\ln(1+x)$ has a power series expansion, then it must be $\ln(1+x) = \displaystyle\sum_{n=0}^{\infty}\frac{(-1)^{n-1}x^n}{n}$.

First we will apply the ratio test to the series to see when this power series converges. Clearly it converges when $x = 0$ because all the terms are zero. So assume $x \ne 0$ and apply the ratio test. Let $a_n = \dfrac{(-1)^n x^n}{n}$. Then

$$\frac{a_{n+1}}{a_n} = \frac{(-1)^{n+1}x^{n+1}/(n+1)}{(-1)^n x^n/n} = -\frac{nx}{n+1}.$$

So,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{n}{n+1}|x| = |x|.$$

Hence the series converges if $|x| < 1$, diverges if $|x| > 1$ and may or may not converge if $|x| = 1$.

So $\ln(1+x)$ cannot have a valid power series expansion when $|x| > 1$. We will not worry about what happens when $|x| = 1$ and will now prove that it has a valid power series expansion when $|x| < 1$. We need to show that if $|x| < 1$, $\lim\limits_{n\to\infty} R_{n,0}(x) = 0$.

Our aim will be to use the Sandwich Theorem to show this. We know that $0 \leq |R_{n,0}(x)|$ so all we need to do is find an upper bound for $|R_{n,0}(x)|$. Taylor's Theorem says that

$$|R_{n,0}(x)| \leq \frac{Mx^{n+1}}{(n+1)!},$$

where M is the absolute maximum of $|f^{(n+1)}(t)|$ on the interval $[0,x]$ or $[x,0]$ (depending on the sign of x). Now,

$$|f^{(n+1)}(t)| = \left|\frac{(-1)^n n!}{(1+t)^{n+1}}\right| \leq \frac{n!}{(1-|t|)^{n+1}}.$$

So the absolute maximum of $|f^{(n+1)}(t)|$ on the interval $[0,x]$ or $[x,0]$ is at most $\dfrac{n!}{(1-|x|)^{n+1}}$. So, for $0 \leq x < 1$, the absolute maximum of $|f^{(n+1)}(t)|$ on the interval $[0,x]$ is $n!$ and we have

$$|R_{n,0}(x) \leq \frac{Mx^{n+1}}{n+1}.$$

So if $0 \leq x < 1$ then $|R_{n,0}(x)| \to 0$ as $n \to 0$.

3. (a) The radius R is $R = \lim\limits_{k\to\infty}\dfrac{k^3}{3^k}\cdot\dfrac{3^{k+1}}{(k+1)^3} = 3\lim\limits_{k\to\infty}\left(\dfrac{k}{k+1}\right)^3 = 3.$

   (b) The radius R is

$$\begin{aligned} R &= \lim_{k\to\infty}\frac{k^k}{(k+1)^{k+1}} = \lim_{k\to\infty}\left(\frac{k}{k+1}\right)^k\cdot\frac{1}{(k+1)} \\ &= \lim_{k\to\infty}\frac{1}{(k+1)}\cdot\lim_{k\to\infty}\left(\frac{1}{1+1/k}\right)^k \\ &= \lim_{k\to\infty}\frac{1}{(k+1)}\cdot\left(\lim_{k\to\infty}(1+1/k)^k\right)^{-1} = \frac{1}{e}\cdot\lim_{k\to\infty}\frac{1}{(k+1)} = 0. \end{aligned}$$

   (c) The radius R is $R = \lim\limits_{k\to\infty}\dfrac{2^k}{k!}\cdot\dfrac{k+1!}{2^{k+1}} = \lim\limits_{k\to\infty}\dfrac{k+1}{2} = +\infty.$

   (d) The radius R is $R = \lim\limits_{k\to\infty}\dfrac{k!}{k+1!} = \lim\limits_{k\to\infty}\dfrac{1}{k+1} = 0.$

   (e) The radius R is $R = \lim\limits_{k\to\infty}\dfrac{k^2}{5^k}\cdot\dfrac{5^{k+1}}{(k+1)^2} = 5\lim\limits_{k\to\infty}\left(\dfrac{k}{k+1}\right)^2 = 5.$

(f) The radius R is $R = \lim_{k\to\infty} \frac{k}{2^k} \cdot \frac{2^{k+1}}{(k+1)} = 2 \lim_{k\to\infty} \frac{k}{k+1} = 2.$

(g) The radius R is

$$R = \lim_{k\to\infty} \frac{(k+1)^{(k+1)}}{k^k} = \lim_{k\to\infty} (k+1) \cdot (1+1/k)^k = e \cdot \lim_{k\to\infty} (k+1) = +\infty.$$

4.

5.

6. We start with the series for e^x: namely $1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \ldots$. Replacing x with $-x^2$, we obtain

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \ldots$$

Now we integrate the series term per term:

$$\begin{aligned} \int_0^1 e^{-x^2}\,dx &= \int_0^1 1 - x^2 + \tfrac{x^4}{2!} - \tfrac{x^6}{3!} + \tfrac{x^8}{4!} - \ldots \\ &= x - \tfrac{x^3}{3} + \tfrac{x^5}{5\cdot 2!} - \tfrac{x^7}{7\cdot 3!} + \tfrac{x^9}{9\cdot 4!} - \ldots \Big|_0^1 \\ &= 1 - \tfrac{1}{3} + \tfrac{1}{5\cdot 2} - \tfrac{1}{7\cdot 3!} + \tfrac{1}{9\cdot 4!} - \ldots \end{aligned}$$

If we add the first three terms here we get .7666. As a rough idea of how accurate this is, suppose we added the next term, this would change the result to .7429. This isn't much of a change. If we added one more term, this would change it even less, the convergence is very slow indeed.

7.

7.7.3 Additional Exercise Sheet 6b

1.

Short Feedback for the Additional Exercise Sheet 6b

1.

7.7.4 Exercise Sheet 0c

1. Compute the Maclaurin series $T_\infty f$ for the following rules defined on the maximal interval containing 0 where they are defined.

   (a) $f(t) = \sinh t$.

   (b) $f(t) = \cosh t$.

   (c) $f(t) = \dfrac{3}{(2-t)^2}$.

   (d) $f(t) = \ln(2+2t)$.

   (e) $f(t) = \dfrac{t}{1-t^2}$.

   (f) $f(t) = \sin t + \cos t$.

   (g) $f(t) = 1 + t^2 - \dfrac{2}{3}t^7$.

   (h) $f(t) = \sqrt[3]{1+t}$.

2. Let $f(t) = \frac{t^4}{1+4t^2}$, what is $f^{(10)}(0)$?

3. Let $f : \mathbb{R} \to \mathbb{R}$ defined by $f(t) = \tan t$.

   (a) What are the first 5 terms of $T_\infty f$?

   (b) Can you find a general formula for the n-th term of $T_\infty f$?

Short Feedback for Exercise Sheet 0c

1. (a) $T_\infty f(t) = \displaystyle\sum_{k=0}^{\infty} \frac{1}{(2k+1)!} t^{2k+1}$.

   (b) $T_\infty f(t) = \displaystyle\sum_{k=0}^{\infty} \frac{1}{(2k)!} t^{2k}$.

   (c) $T_\infty f(t) = \displaystyle\sum_{n=0}^{\infty} \frac{3 \cdot (n+1)}{2^{n+2}} t^n$.

   (d) $T_\infty f(t) = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} t^n$.

   (e) $T_\infty f(t) = \sum_{k=0}^{\infty} t^{2k+1}$.

   (f) $T_\infty f(t) = \displaystyle\sum_{m=0}^{\infty} \left(\frac{t^{4m}}{(4m)!} + \frac{t^{4m+1}}{(4m+1)!} - \frac{t^{4m+2}}{(4m+2)!} - \frac{t^{4m+3}}{(4m+3)!} \right)$.

   (g) $T_\infty f(t) = 1 + t^2 - \dfrac{2}{3}t^7$.

   (h) Use the Binomial Theorem, $T_\infty f(t) = 1 + \displaystyle\sum_{n=1}^{\infty} \frac{(1/3)(1/3-1)(1/3-2)\cdots(1/3-n+1)}{n!} t^n$.

2. $f^{(10)}(0) = 64(10!)$.

3. (a) $T_5 f(t) = t + \frac{1}{3}t^3 + \frac{2}{15}t^5$.

   (b) There is no simple general formula for the n^{th} term in the Taylor series for f.

7.7.5 Feedback for Exercise Sheet 0c

1. (a) $T_\infty f(t) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} t^{2k+1}$.

   (b) $T_\infty f(t) = \sum_{k=0}^{\infty} \frac{1}{(2k)!} t^{2k}$.

   (c) $T_\infty f(t) = \sum_{k=0}^{\infty} \frac{3\cdot(n+1)}{2^{n+2}} t^{n}$.

   (d) $T_\infty f(t) = \sum_{k=1}^{\infty} \frac{(-1)^{n+1}}{n} t^{n}$.

   (e) $\frac{t}{1-t^2}$

   $T_\infty \frac{t}{1-t^2} = T_\infty \left[\frac{1/2}{1-t} - \frac{1/2}{1+t}\right] = t + t^3 + t^5 + \cdots + t^{2k+1} + \cdots$

   (f) $\sin t + \cos t$

   The pattern for the n^{th} derivative repeats every time you increase n by 4. So we indicate the the general terms for $n = 4m, 4m+1, 4m+2$ and $4m+3$:

   $$T_\infty(\sin t + \cos t) = 1 + t - \frac{1}{2!}t^2 - \frac{1}{3!}t^3 + \frac{1}{4!}t^4 + \cdots + \frac{t^{4m}}{(4m)!} + \frac{t^{4m+1}}{(4m+1)!} - \frac{t^{4m+2}}{(4m+2)!} - \frac{t^{4m+3}}{(4m+3)!} + \cdots$$

   (g) $1 + t^2 - \frac{2}{3}t^7$

   A: $T_\infty\left[1 + t^2 - \frac{2}{3}t^7\right] = 1 + t^2 - \frac{2}{3}t^7$

   (h) $\sqrt[3]{1+t}$

   A: $T_\infty \sqrt[3]{1+t} = 1 + \frac{1/3}{1!}t + \frac{(1/3)(1/3-1)}{2!}t^2 + \cdots + \frac{(1/3)(1/3-1)(1/3-2)\cdots(1/3-n+1)}{n!}t^n + \cdots$

2. $f(x) = \frac{x^4}{1+4x^2}$, what is $f^{(10)}(0)$?

   A: $10! \cdot 2^6$

3. $\tan t$ (3 terms only)

   A: $T_3 \tan t = t + \frac{1}{3}t^3$. There is no simple general formula for the n^{th} term in the Taylor series for $\tan x$.

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